A TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY.

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ON

PLANE AND SPHERICAL TRIGONOMETRY,

AND ON

TRIGONOMETRICAL TABLES

AND

LOGARITHMS,

TOGETHER WITH

A SELECTION OF PROBLEMS AND THEIR SOLUTIONS.

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PLANE TRIGONOMETRY.

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TRIGONOMETRY.

SECTION I.

THEORY OF TRIGONOMETRICAL RATIOS.

Object of Trigonometry.—Methods of representing numerically the magnitudes of lines and angles.

1. In any plane triangle there are six parts to be considered, three angles, and three sides. In order to find all the rest, it is in general sufficient to know three of them, but one of the three must be a side; because with three given angles (provided their sum be equal to two right angles) we can form an infinite number of triangles, which are not equal, but only similar to one another. Geometry furnishes simple constructions for each of the cases in which we can determine a triangle by means of some of its parts; but these constructions, on account of the imperfection of the instruments employed, give only a rough and often insufficient approximation. Mathematicians have therefore sought to substitute for them numerical calculations, which always attain the required degree of exactness.

The special object of Trigonometry is to give methods for calculating all the parts of a triangle when there are sufficient data; this is what is called solving a triangle. But in its present enlarged sense, Trigonometry treats of the principles by which angular magnitudes may be estimated, and numerically connected with one another, and with other magnitudes; and shews how to perform measurements generally, by means of the relations of the sides and angles of rectilinear figures.

2. To represent the magnitudes of the sides of a triangle, or of any lines, we refer them to the common unit of length, a foot for instance; and when we represent any line by a general symbol a, we use an abbreviated mode of writing $a \times 1^{ft}$; for a in reality expresses the ratio which the length of the line bears to the assumed unit of length.

3. We represent the magnitudes of angles by numbers expressing how many times they contain a certain angle fixed upon as the unit of angular measure. For this purpose we divide a right angle into 90 equal parts called degrees, each degree into 60 equal parts called minutes, each minute into 60 equal parts called seconds; then an angle is expressed by the number of degrees, minutes, seconds, and decimal parts of a second, which it contains.

Degrees, minutes, and seconds are marked by the symbols $^{\circ}, ', ''$; thus, to represent 14 degrees, 9 minutes, 37,4 seconds, we write $14^{\circ}.9'.37'',4$.

4. Another division of the right angle has sometimes been employed, with a view of assimilating the measures of angles to the decimal notation. In this system the right angle is divided into 100 equal parts called grades, the grade into 100 minutes, the minute into 100 seconds, and so on; and since a minute and second, expressed by decimal parts of a grade, are respectively ·01 and ·0001, an angle expressed in grades, minutes, and seconds, will be represented by the same figures when expressed in grades and decimal parts of a grade; thus $14^{g}.9'.37''$ becomes $14 + 9 \times \cdot01 + 37 \times \cdot0001 = 14\cdot0937$, and expressed in decimal parts of a right angle, it is ·140937.

But notwithstanding this advantage of the centesimal division of the right angle, viz. that an angle given in grades, minutes, &c. can be expressed decimally by inspection, and conversely—whereas, in the sexagesimal division, to effect the same reductions, arithmetical processes are required—the latter division is generally adopted.

5. We shall however here shew how common degrees may be converted into grades, and *vice versâ*.

Let g represent the magnitude of an angle of 1^{g}, and d that of an angle of 1°, and let G, D, be respectively the number of grades and degrees contained in any angle; then

$$Gg = Dd, \quad 100g = 90d; \quad \therefore \frac{G}{100} = \frac{D}{90};$$

$$\therefore G = \frac{10}{9} D = D + \frac{1}{9} D, \text{ and } D = \frac{9}{10} G = G - \frac{1}{10} G.$$

To convert therefore the measure of an angle in degrees to the corresponding measure in grades, we must (after having reduced

the minutes and seconds to the decimal of a degree) increase the measure in degrees by one-ninth of itself, and the result is the measure in grades: to convert, on the other hand, the measure of an angle in grades to the corresponding one in degrees, we must diminish it (expressed decimally) by one-tenth of itself, and the result is the measure in degrees and decimal parts of a degree, which may then be reduced to minutes and seconds.

$$\text{Thus } 29^\circ.5'.33'' = (29{\cdot}0925) \text{ degrees} = \left(29{\cdot}0925 + \frac{29{\cdot}0925}{9}\right) \text{ grades}$$

$$= 32{\cdot}3250 \text{ grades} \quad = 32^g.32'.50''.$$

$$\text{And } 32^g.32'.50'' = (32{\cdot}3250) \text{ grades} = \left(32{\cdot}3250 - \frac{32{\cdot}3250}{10}\right) \text{ degrees}$$

$$= (29{\cdot}0925) \text{ degrees} = 29^\circ.5'.33''.$$

6. Besides the above-mentioned unit of angular measure, viz. the 90th part of a right angle, which is always used in practical applications, there is another, viz. the angle at the centre of a circle which is subtended by an arc equal to the radius of the circle, which is more convenient in analytical investigations.

This angle will be of an invariable magnitude, whatever be the radius of the circle. For, assuming the ratio of the circumference of a circle to its diameter to be invariable, and employing, as usual, the symbol π to express the numerical value of that ratio, so that if r be the radius of the circle, its circumference will equal $2\pi r$, let ACB (fig. 1) be an angle at the centre of a circle subtended by an arc equal to the radius of the circle; then since (Euclid, VI. 33) angles at the centre of the same circle are to one another as the arcs on which they stand,

$$\frac{\text{angle } ACB}{\text{four right angles}} = \frac{\text{arc } AB}{\text{circumference}} = \frac{r}{2\pi r};$$

$$\text{angle } ACB \quad \text{four right angles}$$

which, being independent of r, is invariable.

Since then this angle, which admits of such a simple definition, is of invariable magnitude, it may be properly used to measure other angles. Let it be denoted by ω, then any other angle will be denoted by $\theta\omega$, if θ express the ratio which its

magnitude bears to that of the angle denoted by ω; or, if we choose to suppress the angular unit ω, (in the same manner as we suppress 1^0 when an angle is expressed in degrees,) the angle will be represented simply by θ.

7. In this mode of measuring the magnitudes of angles, θ is equal to the ratio of the arc of a circle subtending the angle, to the radius of the circle, and is therefore called the circular measure.

For let AB, AP (fig. 1) be arcs traced out by any point in the line CP which, revolving from the position AC, describes the angles ACB, ACP; and let arc $AB = AC$, so that ACB is the angle subtended by an arc equal to the radius, or $= \omega$, and ACP any other angle $= \theta\omega$; then (Euclid, VI. 33)

$$\frac{\angle ACP}{\angle ACB} = \frac{AP}{AB} = \frac{AP}{AC},$$

$$\therefore \angle ACP = \angle ACB \cdot \frac{AP}{AC} = \omega \cdot \frac{\text{arc}}{\text{radius}};$$

$$\therefore \theta = \frac{\text{arc}}{\text{radius}}.$$

If we suppose the radius of the circle to be taken equal to the unit of linear measure, we have $\theta = \text{arc}$; or the measure of the angle is the length of the arc subtending it; on this account an angle expressed by its circular measure is sometimes said to be expressed *in arc*.

8. The constant ratio of the circumference of a circle to its diameter, as already stated, is usually denoted by the symbol π; hence the ratio of the semi-circumference to the radius will also be denoted by π, and the ratio of the quadrantal arc to the radius by $\frac{1}{2}\pi$; so that the measures of two right angles, and of a right angle, according to this mode of estimating the magnitudes of angles, will be respectively π and $\frac{1}{2}\pi$. When the fraction arc $\div$ radius is used to measure an angle, we shall generally denote it by a letter of the Greek alphabet; and those, and other symbols, will be used indifferently to designate either the measures of angles, or the angles themselves.

9. Having given the measure of an angle where the 90th part of a right angle is taken for the unit of angular measure, to find its measure where the angle whose arc equals radius is taken for the unit, and conversely.

The numerical value of π, the ratio of the circumference of a circle to its diameter, is 3·14159. Let r be the radius and c the circumference of a circle, then $c = 2\pi r$; also let x be the number of degrees in ω, the angle whose arc equals radius, then (Euclid, VI. 33),

$$\frac{x^0}{180^0} = \frac{r}{\pi r};$$

$$\therefore\ x^0 = \frac{180^0}{3{\cdot}14159} = 57^0{\cdot}29577.$$

Hence if an angle θ be given referred to the unit ω, its measure in degrees will be $\theta \times 57^0{\cdot}29577$; and, conversely, if an angle A be given in degrees, its measure when ω is taken for the unit, will be $\dfrac{A}{57{\cdot}29577}$.

Definitions of the Trigonometrical Ratios.

10. In trying to connect numerically the angles and the sides of triangles, our first idea would be to find direct relations between the sides and those ratios, involving the lengths of circular arcs, which serve as the measures of the angles. But as circular arcs cannot be compared with one another or with straight lines, by means of their geometrical properties, we soon become aware of the difficulties of introducing the measures of the angles into calculation; and we are led to replace them by certain ratios depending upon the angles, so that they are determined when the angles are known, and conversely; and which, involving the sides of right-angled triangles only, are capable of comparison with one another by means of their geometrical properties. These ratios, the use of which now extends to all branches of Mathematics, are what are called collectively Trigonometrical Ratios. We proceed to give definitions of them.

11. If from any point in either of the indefinitely produced sides containing an angle, a perpendicular be dropped

upon the other side, or the other side produced backwards, so forming a right-angled triangle, then

The ratio of the side opposite to the angle, to the hypothenuse, or the fraction $\frac{\text{perpendicular}}{\text{hypothenuse}}$, is called the Sine of the angle.

The ratio of the side opposite to the angle, to the side adjacent to it, or the fraction $\frac{\text{perpendicular}}{\text{base}}$, is called the Tangent of the angle.

The ratio of the hypothenuse to the side adjacent to the angle, or the fraction $\frac{\text{hypothenuse}}{\text{base}}$, is called the Secant of the angle.

Thus (fig. 2) if the line AC revolving from the position AB describe the angle BAC containing A degrees; and if from any point P in AC we drop a perpendicular PN upon AB, or AB produced backwards, we have, employing the usual abbreviated mode of writing the Trigonometrical Ratios,

$$\sin A = \frac{PN}{AP}, \quad \tan A = \frac{PN}{AN}, \quad \sec A = \frac{AP}{AN}.$$

12. The Complement of an angle is that angle which must be added to it to make a right angle; thus the complement of 45^0 is 45^0, and the complement of 30^0 is 60^0. When the angle is greater than 90^0, its complement is negative; thus the complement of 127^0 is -37^0. The two acute angles of a right-angled triangle are complements one of the other.

The Cosine, Cotangent, and Cosecant of an angle, are the sine, tangent, and secant of its complement, and are denoted by the abbreviations cos, cot, cosec. Hence, according to these definitions,

$$\cos A = \sin(90^0 - A), \ \cot A = \tan(90^0 - A), \ \operatorname{cosec} A = \sec(90^0 - A).$$

Also, referring to fig. 2,

$$\cos A = \sin APN = \frac{AN}{AP} = \frac{\text{base}}{\text{hypothenuse}},$$

or the Cosine of the angle A, is the ratio of the side adjacent to the angle A, to the hypothenuse;

$$\cot A = \tan APN = \frac{AN}{PN} = \frac{\text{base}}{\text{perpendicular}},$$

$$\operatorname{cosec} A = \sec APN = \frac{AP}{PN} = \frac{\text{hypothenuse}}{\text{perpendicular}}.$$

In conformity with the definitions we likewise have

$$\cos(90^0 + A) = \sin(-A),\quad \cot(90^0 + A) = \tan(-A),$$

$$\operatorname{cosec}(90^0 + A) = \sec(-A).$$

13. Hence we see that the cosecant, cotangent, and cosine of an angle, are respectively equal to the reciprocals of the sine, tangent, and secant of the angle, so that we have

$$\operatorname{cosec} A = \frac{1}{\sin A},\quad \cot A = \frac{1}{\tan A},\quad \cos A = \frac{1}{\sec A};$$

and that, consequently, the cosecant, cotangent, and cosine might have been defined to be the reciprocals of the sine, tangent, and secant.

As however the sine, cosine, and tangent are by far the most frequently used, they must be regarded as forming the primary class of the Trigonometrical Ratios; and the others as forming a subordinate class, the employment of which is occasionally attended with conveniences which will be hereafter pointed out.

There are also two other quantities which are sometimes employed to determine an angle A, viz. versine A, and su-versine A; they are used to express $1 - \cos A$ and $1 + \cos A$.

14. The Trigonometrical Ratios determine the angles, and conversely; i. e. any determinate values being given for the one, determinate values can be found for the other.

For suppose the angle to be given, and that it is BAC (fig. 3); then as long as the angle continues the same, the values of the Trigonometrical Ratios remain the same, wherever the point P is taken in AC; for if we take a second point P' and drop a perpendicular $P'N'$, since the triangles APN,

$AP'N'$ are similar, their sides have to one another the same ratios, and therefore $\sin A$, $\tan A$, &c. will have the same values, whether APN or $AP'N'$ be the triangle by the sides of which they are expressed.

Again, suppose the values of any of the Trigonometrical Ratios given, $\sin A = a$, $\cos A = a$, $\tan A = a$, for instance. Taking any line CP (fig. 4) describe upon it as diameter a semi-circle, and from centre P with a radius which is to CP as a to 1, describe a circle cutting the former in N, and join CN, PN; then PCN is an angle whose sine equals a; for on account of the right angle PNC,

$$\sin PCN = \frac{PN}{CP} = a.$$

If it is the cosine which is given, the construction will be the same, except that the second circle will be described from centre C with a radius CN which is to CP as a to 1.

Again, taking any line AN (fig. 2) erect PN perpendicular to it, and having to AN the same ratio that a has to 1; join AP, then PAN is an angle whose tangent equals a, for $\tan PAN = \dfrac{PN}{AN} = a$.

These geometrical solutions of the problem of finding an angle from its sine, tangent, &c. are given only by way of illustration; we shall hereafter shew how to find the values of the Trigonometrical Ratios of all angles to any degree of accuracy; and if these values and the angles be arranged in Tables, side by side, they will mutually determine one another with a precision unattainable by geometrical constructions.

15. From the above considerations we also see that for the same angle, there is one determinate value and only one of each of its Trigonometrical Ratios. The converse Proposition is not true, viz. that, corresponding to a given value of the sine, tangent, &c. there is only one determinate value of the angle. On the contrary, we shall see further on, that, allowing our definitions their necessary generality, there is, corresponding to the same value of the sine, tangent, &c. an indefinite number of values of the angle.

Relations of the Trigonometrical Ratios to one another.

16. To express the cosine of an angle by means of its sine, and *vice versâ*; and to express the other Trigonometrical Ratios of an angle by means of its sine and cosine.

From the right-angled triangle APN (fig. 2)

$$PN^2 + AN^2 = AP^2;$$

$$\therefore \left(\frac{PN}{AP}\right)^2 + \left(\frac{AN}{AP}\right)^2 = 1,$$

$$\text{or } \sin^2 A + \cos^2 A = 1 \text{......}(1);$$

$$\therefore \cos A = \pm\sqrt{1 - \sin^2 A}, \text{ and } \sin A = \pm\sqrt{1 - \cos^2 A}.$$

$$\text{Also } \tan A = \frac{PN}{AN} = \frac{\frac{PN}{AP}}{\frac{AN}{AP}} = \frac{\sin A}{\cos A} \text{......}(2)$$

$$\text{and, by Art. 13,} \quad \sec A = \frac{1}{\cos A} \text{......}(3)$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A} \text{......}(4)$$

$$\operatorname{cosec} A = \frac{1}{\sin A} \text{..............} (5).$$

17. The four latter formulæ enable us to find the values of $\tan A$, $\sec A$, &c. when those of $\sin A$ and $\cos A$ are known.

Ex. 1. Suppose $A = 45^0$.

Let ABC (fig. 5) be an isosceles triangle, right-angled at C; then the other angles are equal, and their sum is a right angle, therefore each of them is half a right angle.

Now $AB^2 = AC^2 + BC^2 = 2BC^2$, or $AB = \sqrt{2} \,.\, BC$;

$$\therefore \sin 45^0 = \frac{BC}{AB} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\cos 45^0 = \sin(90^0 - 45^0) = \sin 45^0 = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45^0 = \cot 45^0 = 1,$$

$$\sec 45^0 = \operatorname{cosec} 45^0 = \sqrt{2}.$$

Ex. 2. Suppose $A = 30^0$.

Let ABC (fig. 6) be an equilateral triangle; therefore each of its angles $= \frac{1}{3}$rd of two right angles $= 60^0$.

Draw BD a perpendicular from any angle upon the opposite side; then BD bisects both the angle ABC, and the side AC.

$$\therefore \sin ABD = \frac{AD}{AB} = \frac{\frac{1}{2}AB}{AB} = \tfrac{1}{2};$$

$$\text{or } \sin 30^0 = \cos 60^0 = \tfrac{1}{2},$$

$$\cos 30^0 = \sin 60^0 = \sqrt{1 - \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{3},$$

$$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3},$$

$$\cot 30^0 = \tan 60^0 = \sqrt{3},$$

$$\sec 30^0 = \operatorname{cosec} 60^0 = \tfrac{2}{3}\sqrt{3},$$

$$\operatorname{cosec} 30^0 = \sec 60^0 = 2.$$

These values may of course be obtained, not through the sine, but directly from the figure; for

$$AD = \tfrac{1}{2}AB, \text{ and } DB^2 = AB^2 - AD^2 = \tfrac{3}{4}AB^2,$$

$$\text{or } DB = \tfrac{1}{2}\sqrt{3}\,.\,AB;$$

$$\therefore \cos 30^0 = \frac{DB}{AB} = \tfrac{1}{2}\sqrt{3}, \ \tan 30^0 = \frac{AD}{DB} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}, \ \&c.$$

18. The five formulæ of Art. 16 are the fundamental ones, and will enable us to deduce all others in which only one angle is involved; the following are the most remarkable.

Squaring, and adding unity to each side of (2), we have

$$1 + \tan^2 A = 1 + \frac{\sin^2 A}{\cos^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A},$$

$$\text{but } \sin^2 A + \cos^2 A = 1, \text{ and } \frac{1}{\cos^2 A} = \sec^2 A;$$

$$\therefore 1 + \tan^2 A = \sec^2 A.$$

This also appears from triangle APN (fig. 2) in which

$$AP^2 = AN^2 + NP^2;$$

$$\therefore \left(\frac{AP}{AN}\right)^2 = 1 + \left(\frac{PN}{NA}\right)^2,$$

$$\text{or } \sec^2 A = 1 + \tan^2 A.$$

Similarly,
$$\left(\frac{AP}{PN}\right)^2 = 1 + \left(\frac{AN}{PN}\right)^2,$$

$$\text{or } \operatorname{cosec}^2 A = 1 + \cot^2 A,$$

which also results from the preceding by writing $90^0 - A$ for A.

19. In general, any one of the six Trigonometrical Ratios of an angle being given, the five fundamental relations will enable us to find the values of all the rest; and for that purpose it will only be requisite to effect the simple solution of equations.

Suppose, for instance, it was required to find the sine and cosine of an angle by means of its tangent, we must take the equations (1) and (2), viz.

$$\sin^2 A + \cos^2 A = 1, \quad \tan A = \frac{\sin A}{\cos A};$$

the second gives $\sin A = \tan A \cos A$, or $\sin^2 A = \tan^2 A \cos^2 A$; and substituting this value of $\sin^2 A$ in the former, we easily find

$$\cos A = \pm \frac{1}{\sqrt{1 + \tan^2 A}}, \text{ and then } \sin A = \pm \frac{\tan A}{\sqrt{1 + \tan^2 A}};$$

the double sign implying that there exist two sines and two cosines, equal in magnitude but contrary in algebraical sign, corresponding to the same tangent; the explanation will be found in a subsequent Article.

Use of the signs + and − to indicate contrariety of position.

20. In giving the line AC (fig. 2) which, revolving about A from AB, describes the angle BAC, all possible positions, the sides of the determining triangle may assume situations entirely contrary to those which they have when the angle is less than 90^0. For instance, in the case of the angle BAC greater than 90^0, the base AN is situated to the left of A, while before it was situated to the right.

These contrarieties of position may be taken account of in our calculations, by affecting the quantities representing the magnitudes of the lines with contrary algebraical signs.

For let BC be any line (fig. 7), in which are given two points A, B, separated by a distance $AB = a$; and suppose the distance from A of a point N in the line to be known, and $= b$, and we wish to express the distance of N from B; if we denote this distance by x, we have

$$x = a + b, \quad \text{or } x = a - b,$$

according as N lies in AC or AB; so that it will be necessary to use two formulæ for these two positions of N. But this inconvenience may be eluded, and a single formula will suffice, if we take care to give different algebraical signs to distances which have contrary situations with respect to the origin A. In fact, the first formula $x = a + b$, if we suppose b to be negative, becomes $x = a - b$, which corresponds to a point N' placed just as far to the left of A as N is to the right, and so will serve to determine all positions of N in the indefinite line BC, or all distances of N from any line AD, measured parallel to BC.

Similarly, if distances originating in BC, and taken along AD or only parallel to AD, be denoted by positive quantities when they are measured upwards, when measured downwards they will be denoted by negative quantities. As the signs + and − are not sufficient to express the relation of position of lines to each other which are inclined at any angle to one another, they are only applicable to lines which are either in the same straight line, or are parallel to one another; but we shall soon see how, by the employment of Trigonometrical Ratios, the more general relation of position likewise may be expressed.

21. Again, we may bring angles under the same rule; for let AB, AD (fig. 8) be two lines including a known angle A, and suppose the inclination of a line AC to AB to be known, and $= B$; and we wish to express its inclination to AD; calling this latter angle C, we have

$$C = A + B, \quad \text{or } C = A - B,$$

according as AC falls above or below AB; both of which for-

mulæ may be included in a single one, $C = A + B$, if we affect B with a positive or negative sign, according as the angle which it represents is formed above or below the line AB.

Hence if angles described by a line revolving from a fixed position in one direction be denoted by positive quantities, those which are described by the line revolving from the same initial position in the opposite direction will be denoted by negative quantities; and the arcs which are the measures of the angles will be affected with their proper algebraical signs according to the same rule.

22. By these considerations we are led to the following important principle, first established by *Des Cartes*.

If on any line, straight or curved, we consider different distances measured from a fixed point in the line as the common origin, we shall introduce into the calculation the distances which have contrary situations relative to the origin, by affecting the one with the sign +, and the other with the sign −.

The direction of the positive distances is quite indifferent; but being once fixed, the negative distances must lie in the contrary direction. With respect to the sides opposite and adjacent to an angle in the determining triangle, and which, for different angles measured from the same primitive line, are always either parallel, or in the same straight line, it is usual to consider them positive in the situations which they occupy when the angle is less than 90^o. With respect to the hypothenuse, it is necessary to take only its magnitude into account; for its position is fixed by the other two sides, and cannot be determined by the signs + and −, since it does not remain parallel to a fixed line.

This principle of *Des Cartes* is not to be assimilated to a theorem capable of being demonstrated *à priori*; it is in truth but a simple convention, which we must be careful not afterwards to contradict, and of which the utility is rendered evident by the applications we make of it.

23. The Supplement of an angle is that angle which must be added to it to make two right angles; when the angle is greater than 180^0, its supplement is negative; thus the supplement of 100^0 is 80^0, and the supplement of 200^0 is -20^0.

If two angles be supplementary to one another, their Trigonometrical Ratios are all equal and of contrary signs, with

the exception of the sines and cosecants, which are equal and of the same sign.

Let BAC (fig. 9) be an angle greater than a right angle, containing A degrees. In AC take any point Q, and draw QM perpendicular to BA produced; also make angle $BAC' = \angle QAM$, $AP = AQ$, and draw PN perpendicular to AB. Then the triangles PAN, QAM, are equal in all respects; and we may assume $AN = b$, $PN = p$, b and p being positive quantities, and $AP = r$; therefore $AM = -b$, $MQ = p$, $AQ = r$;

$$\text{now } \sin BAC = \sin A = \frac{QM}{AQ} = \frac{p}{r},$$

$$\sin BAC' = \sin (180^0 - A) = \frac{PN}{AP} = \frac{p}{r};$$

$$\therefore \sin A = \sin (180^0 - A).$$

$$\text{Also, } \cos BAC = \cos A = \frac{AM}{AQ} = \frac{-b}{r},$$

$$\cos BAC' = \cos (180^0 - A) = \frac{AN}{AP} = \frac{b}{r};$$

$$\therefore \cos A = -\cos (180^0 - A).$$

Hence

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin (180^0 - A)}{-\cos (180^0 - A)} = -\tan (180^0 - A),$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{-\cos (180^0 - A)} = -\sec (180^0 - A),$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{-\tan (180^0 - A)} = -\cot (180^0 - A),$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{\sin (180^0 - A)} = \operatorname{cosec} (180^0 - A).$$

24. If two angles are equal but of contrary signs, then all their Trigonometrical Ratios are equal and of contrary signs, with the exception of the cosines and secants, which are equal and of the same sign.

Let AC (fig. 10) revolving about the point A from the initial position AB, describe the angle BAC containing A de-

grees; then if it revolve in the contrary direction from AB till angle $BAC' = BAC$, we shall have angle $BAC' = -A$. Through any point N in AB draw PNQ perpendicular to AB and meeting AC, AC', in P and Q; and let $AP = r$, $AN = b$, $PN = p$; then $NQ = -p$, and $AQ = r$.

Now $$\sin BAC = \sin A = \frac{PN}{AP} = \frac{p}{r},$$

$$\sin BAC' = \sin(-A) = \frac{QN}{AQ} = \frac{-p}{r};$$

$$\therefore \sin A = -\sin(-A).$$

Again, $$\cos BAC = \frac{AN}{AP} = \frac{b}{r},$$

$$\cos BAC' = \frac{AN}{AQ} = \frac{b}{r};$$

$$\therefore \cos A = \cos(-A).$$

Also $$\tan A = \frac{\sin A}{\cos A} = \frac{-\sin(-A)}{\cos(-A)} = -\tan(-A),$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\cos(-A)} = \sec(-A),$$

$$\cot A = -\cot(-A), \quad \text{cosec}\, A = -\text{cosec}\,(-A).$$

Magnitudes of angles unlimited.—Method of reducing the Trigonometrical Ratios of all angles to those of angles less than a right angle.

25. Every angle considered in Geometry is less than two right angles; but in Trigonometry the term *angle* has a far wider signification; and the representation of angles by means of their measures, leads to the consideration of angles not only greater than two right angles, but of all possible magnitudes.

With centre A (fig. 11) and any radius AB describe a circle; and suppose the radius AC, starting from the initial position AB, to revolve in the direction BCC_1 and so to form different positive angles $BAC = \theta$, $BAC_1 = \theta_1$, with AB. Then if AC continue to revolve in the same direction, we shall have the

angle $\pi + B'AC_2$ greater than two right angles, with measure $\theta_2 = \frac{BDB'C_2}{AB}$, and the angle $\pi + B'AC_3$ greater than three right angles, with measure $\theta_3 = \frac{BDB'C_3}{AB}$. When the describing radius reaches AB, the whole angle described is four right angles or 2π; and when it reaches AC for the second time, the whole angle described is $2\pi + BAC$ or $2\pi + \theta$; and for the n^{th} time, the whole angle described is $2(n-1)\pi + \theta$.

Similarly, negative angles of all magnitudes may be formed by the describing line revolving from AB in a direction opposite to its former one; and the various positions of AC relative to the primitive line will be equally well determined by them as by positive angles. Thus the positions of AC_2, AC_3, may be determined by the negative angles BAC_2, BAC_3, that is, by $-(2\pi - \theta_2)$, $-(2\pi - \theta_3)$, as well as by θ_2, θ_3.

*26. To trace the changes in the magnitudes and algebraic signs of the Trigonometrical Ratios of an angle, as the angle increases from zero to 2π.

Since the values of the Trigonometrical Ratios of an angle are independent of the magnitude of the hypothenuse of the determining triangle, we may suppose it to preserve the same constant value $= r$; and we shall consider four cases according as the angle lies between zero and a right angle, between one right angle and two, between two and three, or between three and four right angles.

I. When the describing radius AC (fig. 2) coincides with AB, and the angle BAC consequently is zero, PN vanishes, and AN becomes equal to AP or r; hence

$$\sin 0 = \frac{0}{r} = 0, \quad \cos 0 = \frac{r}{r} = 1, \quad \tan 0 = \frac{0}{r} = 0;$$

$$\therefore \operatorname{cosec} 0 = \infty, \quad \sec 0 = 1, \quad \cot 0 = \infty.$$

As the angle increases, PN increases and AN diminishes; therefore the sine, tangent, and secant increase, and the cosine, cotangent, and cosecant diminish; till the angle becomes a right angle when PN becomes equal to AP or r, and AN vanishes;

$$\therefore \sin\frac{\pi}{2} = \frac{r}{r} = 1, \quad \cos\frac{\pi}{2} = \frac{0}{r} = 0, \quad \tan\frac{\pi}{2} = \frac{r}{0} = \infty,$$

$$\operatorname{cosec}\frac{\pi}{2} = 1, \quad \sec\frac{\pi}{2} = \infty, \quad \cot\frac{\pi}{2} = 0.$$

All the Trigonometrical Ratios of an angle lying between zero and a right angle, are positive.

II. When the angle BAC increases from $\frac{1}{2}\pi$ to π (fig. 2) PN diminishes and AN increases; therefore the absolute values of the sine, tangent, and secant diminish, whilst those of the cosine, cotangent, and cosecant, increase. Also since the value of PN is positive and that of AN negative, the values of the sine and cosecant are positive, and those of the rest of the Trigonometrical Ratios negative. When the angle increases up to π, so that PN vanishes, and $AN = -r$, we have

$$\sin\pi = \frac{0}{r} = 0, \quad \cos\pi = \frac{-r}{r} = -1, \quad \tan\pi = \frac{0}{-r} = 0,$$

$$\operatorname{cosec}\pi = \infty, \quad \sec\pi = -1, \quad \cot\pi = \infty.$$

III. Similarly, as the angle BAC (fig. 12) increases from π to $\frac{3\pi}{2}$, the sine, tangent, and secant increase, and the other Trigonometrical Ratios diminish; and as the values both of PN and AN are negative, the sine, cosine, cosecant, and secant are negative, but the tangent and cotangent positive; and when the angle increases up to $\frac{3\pi}{2}$, so that AN vanishes, and $PN = -r$, we have

$$\sin\frac{3\pi}{2} = \frac{-r}{r} = -1, \quad \cos\frac{3\pi}{2} = \frac{0}{r} = 0, \quad \tan\frac{3\pi}{2} = \frac{-r}{0} = -\infty,$$

$$\operatorname{cosec}\frac{3\pi}{2} = -1, \quad \sec\frac{3\pi}{2} = -\infty, \quad \cot\frac{3\pi}{2} = 0.$$

IV. Lastly, as the angle BAC (fig. 13) increases from $\frac{3\pi}{2}$ to 2π, the sine, tangent, and secant diminish, and the rest increase; and as the value of PN is negative, but that of AN

positive, the cosine and secant are positive, and the rest negative; and when the angle becomes 2π, we have

$$\sin 2\pi = \frac{0}{r} = 0, \quad \cos 2\pi = \frac{r}{r} = 1, \quad \tan 2\pi = \frac{0}{r} = 0,$$

$$\operatorname{cosec} 2\pi = \infty, \quad \sec 2\pi = 1, \quad \cot 2\pi = -\infty.$$

Hence we conclude,

1. The sines and cosines of angles are always less than unity, and may represent all proper fractions positive or negative, but no other numbers.

2. It is only the tangent and cotangent that are continuous through all values from positive to negative infinity, and may represent all real numbers whatever.

3. The secant and cosecant are discontinuous from $+1$, to -1, but may represent any number whatever that is not a proper fraction.

27. In the applications of Analysis, as has been stated, we have frequently to consider angles which contain π several times: we must therefore give formulæ for expressing the Trigonometrical Ratios of all such angles by those of other angles less than $\frac{\pi}{2}$. We shall especially consider the Sine and Cosine, which are the Ratios most used; and as every angle greater than π must consist of an angle $< \pi$, together with π, once or several times repeated, we shall first examine what would be the sine and cosine of $\pi + \theta$, θ being less than π.

Let the angle BAC (fig. 14) be denoted by θ; produce CA to C', then the positive angle BAC' will be denoted by $\pi + \theta$; and these angles will have equal sines $\frac{PN}{AP}$, $\frac{P'N'}{AP'}$; but as the lines PN, $P'N'$ have contrary positions, they must be affected with different algebraic signs; the cosines in like manner $\frac{AN}{AP}$, $\frac{AN'}{AP'}$, are equal and must be affected with contrary algebraic signs;

$$\therefore \sin(\pi + \theta) = -\sin\theta, \quad \cos(\pi + \theta) = -\cos\theta.$$

Next suppose θ increased by 2π, then the line AC will return to its original position, and all the Trigonometrical Ratios will remain the same;

$$\therefore \sin(2\pi + \theta) = \sin\theta; \quad \cos(2\pi + \theta) = \cos\theta.$$

In general, whatever be the magnitude of the angle θ, if we add to it π, or any odd multiple of π, the bounding radius will be transported to a position exactly opposite to that which it first occupied, and then, as is evident, the sine and cosine have only their algebraic signs altered, but not their magnitudes; but if we add to it 2π, or any even multiple of π, the bounding radius returns to its original position, and the Trigonometrical Ratios are altered neither in magnitude nor algebraic sign. Hence the sine of any multiple of π will be zero; and the cosine will equal $+1$ or -1 according as the multiple is even or odd: i. e. $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.

From the preceding results we readily perceive that

$$\tan(\pi + \theta) = \tan\theta, \quad \tan(2\pi + \theta) = \tan\theta.$$

28. It is proper to observe, that the formulæ proved here, and at Arts. 23, 24,

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta \ldots (1)$$

$$\sin(\pi + \theta) = -\sin\theta, \qquad \cos(\pi + \theta) = -\cos\theta \ldots (2)$$

$$\sin(2\pi + \theta) = \sin\theta, \qquad \cos(2\pi + \theta) = \cos\theta \ldots\ldots (3)$$

$$\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta \ldots\ldots (4)$$

are true for positive and negative angles of all magnitudes.

The first were proved only for values of θ between zero and π; changing θ into $\pi + \theta$, they become

$$\sin(-\theta) = \sin(\pi + \theta), \quad \cos(-\theta) = -\cos(\pi + \theta),$$

which are evidently true by (2) and (4). We may now again increase θ by π, and so on to any extent; also putting $-\theta$ instead of θ, we see that the two formulæ are still true; therefore they hold for all angles whatever.

Formulæ (2) we have seen to be true for all positive angles; also if we replace θ by $-\theta$, they become identical with (1), therefore they subsist also for negative angles.

In formulæ (4) it is evident that θ may be replaced by $-\theta$; and from the way in which these formulæ are established, there is no limitation to the magnitude of θ.

Formulæ (3) we have seen to be true for all positive values of θ; and since the addition of 2π to any angle, positive or negative, makes no alteration in its sine or cosine, they must be true for negative values of θ.

Hence also, from Art. 12, we have

$$\cos\left(\frac{\pi}{2}+\theta\right) = \sin(-\theta) = -\sin\theta,$$

$$\cot\left(\frac{\pi}{2}+\theta\right) = \tan(-\theta) = -\tan\theta,$$

$$\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec(-\theta) = +\sec\theta.$$

29. It is now easy to reduce the Trigonometrical Ratios of any angle whatever, to those of an angle less than 90^0.

We must first suppress 360^0 as often as we can, and the sine, cosine, and tangent remain unaltered. We must next suppress 180^0 (if the angle exceed 180^0) and change the signs of sine and cosine, but not of tangent. If the angle which now remains be greater than 90^0, we must take its supplement, and change the algebraic signs of the cosine and tangent, but not of the sine. The values of the other Trigonometrical Ratios may be found by expressing them by the sine, cosine, or tangent.

Ex. $\sin 1029^0 = \sin 309^0 = -\sin 129^0 = -\sin 51^0.$

$$\cos 1029^0 = \cos 309^0 = -\cos 129^0 = \cos 51^0.$$

$$\tan 675^0 = \tan 315^0 = \tan 135^0 = -\tan 45^0.$$

$$\sin(-1029^0) = \sin(-309^0) = -\sin(-129^0) = \sin 51^0.$$

30. Since $\sin\theta = \sin(\pi-\theta) = -\sin(-\theta) = -\sin(\pi+\theta)$, and since we are at liberty to add or subtract any multiple of 2π to or from an angle without altering the values of its Trigonometrical Ratios, we have

$$\sin\theta = \sin(2n\pi + \theta) = \sin\{(2n+1)\pi - \theta\} = -\sin(2n\pi - \theta)$$
$$= -\sin\{(2n+1)\pi + \theta\};$$

or, expressed by a single formula, $\sin\theta = \sin\{n\pi + (-1)^n\theta\}$.

Similarly,

$$\cos\theta = \cos(2n\pi + \theta) = -\cos\{(2n+1)\pi - \theta\} = \cos(2n\pi - \theta)$$
$$= -\cos\{(2n+1)\pi + \theta\};$$

or, expressed by a single formula, $\cos\theta = (-1)^n \cos(n\pi \pm \theta)$.

And since $\tan\theta = -\tan(\pi - \theta) = -\tan(-\theta) = \tan(\pi + \theta)$,

$$\tan\theta = \tan(2n\pi + \theta) = -\tan\{(2n+1)\pi - \theta\} = -\tan(2n\pi - \theta)$$
$$= \tan\{(2n+1)\pi + \theta\};$$

or, expressed by a single formula, $\tan\theta = \tan(n\pi + \theta)$.

In all these expressions n is zero or any positive or negative integer; and θ is any angle positive or negative. Similar formulæ may of course be obtained for the other Trigonometrical Ratios.

On the angles which correspond to given values of the sine, cosine, &c.

31. The preceding results give occasion for the important remark that there exists an infinite number of angles which have the same Trigonometrical Ratios. We will now therefore suppose the values of some of those ratios to be given, and determine all the angles which correspond to them.

To find all the values of the angle θ which satisfy the equation $\sin\theta = a$.

Construct, as in Art. 14, the angle $BAC' = \alpha$, (fig. 9), having its sine $= a$; and let $BAC = \pi - \alpha$ be the supplement of BAC'. Then the equation $\sin\theta = a$ can be satisfied only by angles which are bounded by AB and AC', or by AB and AC; hence the positive angles will be BAC', and BAC' increased by any multiple of 2π; and BAC, and BAC increased by any multiple of 2π; i.e. they will be

$$2n\pi + \alpha, \text{ and } 2n\pi + (\pi - \alpha);$$

and the negative angles will be BAC', BAC, reckoned in the

negative order, and the sums of each of these angles and any multiple of 2π taken negatively, i.e. they will be

$$-2n\pi-(2\pi-a),\ \text{and}\ -2n\pi-(\pi+a),$$

$$\text{or}\ -2(n+1)\pi+a,\ \text{and}\ -(2n+1)\pi-a;$$

both of which series of angles are comprised in the expression

$$n\pi+(-1)^n a,$$

n being any positive or negative integer not excluding zero, which, consequently, is the general value of θ, or the general form of equisinal angles.

32. To find all the values of the angle θ which satisfy the equation $\cos\theta = a$.

Construct the angle $BAC = a$ (fig. 10) having its cosine equal to a, and make the negative angle $BAC' = BAC$. Then the equation $\cos\theta = a$ can be satisfied only by angles which are bounded by AB and AC, or by AB and AC'. Hence the positive angles will be BAC, and BAC increased by any multiple of 2π; and BAC' (reckoned in the positive order), and BAC' increased by any multiple of 2π; i.e. they will be

$$2n\pi+a,\ \text{and}\ 2n\pi+(2\pi-a);$$

and the negative angles will be BAC, BAC', reckoned in the negative order, and the sums of each of these angles and any multiple of 2π taken negatively, i.e. they will be

$$-2n\pi-(2\pi-a),\ \text{and}\ -2n\pi-a;$$

both of which series of angles are comprised in the expression

$$2n\pi\pm a,$$

n being any positive or negative integer not excluding zero, which, consequently, is the general value of θ.

33. To find all the values of the angle θ which satisfy the equation $\tan\theta = a$.

As in Art. 14, construct the angle $BAC = a$ (fig. 14) having its tangent equal to a, and produce CA to C'. Then the equation $\tan\theta = a$ can be satisfied only by angles which are bounded by AB and AC, or by AB and AC'. Hence the

positive angles will be BAC, and BAC increased by any multiple of 2π; and BAC' (reckoned in the positive order), and BAC' increased by any multiple of 2π; i. e. they will be

$$2n\pi + a, \text{ and } 2n\pi + (\pi + a);$$

and the negative angles will be BAC, BAC', reckoned in the negative order, and the sums of each of these angles and any multiple of 2π taken negatively, i. e. they will be

$$-2n\pi - (2\pi - a), \text{ and } -2n\pi - (\pi - a),$$

$$\text{or } -2(n+1)\pi + a, \text{ and } -(2n+1)\pi + a;$$

both of which series of angles are comprised in the expression $n\pi + a$, n being any positive or negative integer not excluding zero, which, consequently, is the general value of θ.

Since (Art. 30) $\cos(n\pi + a) = \pm\cos a$, $\sin(n\pi + a) = \pm\sin a$, according as n is even or odd, it appears that if we express the cosine and sine of an angle by its tangent as in Art. 19, we must obtain for each two values and no more, equal in magnitude but contrary in algebraic sign.

34. It is unnecessary to go through the cases in which the angle is given by any other of its Trigonometrical Ratios; for since

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \text{and } \cot\theta = \frac{1}{\tan\theta},$$

the formulæ for the solution of $\operatorname{cosec}\theta = a$, $\sec\theta = a$, $\cot\theta = a$, will be exactly the same as those for $\sin\theta = a$, $\cos\theta = a$, $\tan\theta = a$. It must not be forgotten that in the equations $\sin\theta = a$, $\cos\theta = a$, a must lie between $+1$ and -1, and that in the equation $\tan\theta = a$, a lies between $+\infty$ and $-\infty$; also, that in the above formulæ, a is the least positive angle which corresponds to the given Trigonometrical Ratio, and therefore is always intermediate to zero and π; unless a negative value of the sine or cosecant be given, in which case the least corresponding angle will lie between π and $3\pi \div 2$.

Obs. It was observed that, besides the ordinary signs $+$ and $-$ which are used to indicate contrariety of position, there is a general sign of affection, viz. $\cos\theta + \sqrt{-1}\sin\theta$, which expresses the position of a line inclined to the initial line; so

that $a\,(\cos\theta + \sqrt{-1}\sin\theta)$ represents in magnitude and position, a line whose length is a, and which makes an angle θ with the initial line. For it is proved (Theory of Equations, Art. 22) that if

$$\alpha = \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n},$$

then $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ are the roots of $x^n - 1 = 0$; and if the period be continued, the roots will recur perpetually in the same order. Now let there be drawn in a circle n radii $a, a_1, a_2, \ldots a_{n-1}$ making equal angles $2\pi \div n$ with one another; then the relative positions of these lines will present an exact correspondence with the conditions fulfilled by the symbolical quantities, a, $a\alpha$, $a\alpha^2, \ldots a\alpha^{n-1}$; and the radii may, consequently, be represented in magnitude and position by those quantities, which, though symbolically different, are all of the same magnitude, α being one of the roots of unity. For if we take $a\alpha$ to represent a_1, since a_2 is in the same position relative to a_1 that a_1 is in relative to a, we must have $(a\alpha)\,\alpha$ or $a\alpha^2$ to represent a_2; similarly, $a\alpha^3$ will represent in magnitude and position a_3, and so on. So that the lines a_1, a_2, &c. which are assumed to be represented with respect to each other by $a\alpha$, $a_1\alpha$, &c., will be represented relative to the primitive line a, by the several terms $a\alpha$, $a\alpha^2 \ldots a\alpha^{n-1}$, $a\alpha^n$; the last term coinciding with the primitive line a conformably to the condition $\alpha^n = 1$. Hence, generally,

$$a\,\alpha^r = a\left(\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\right).$$

will represent a_r the radius inclined at an angle $2r\pi \div n$, to the primitive; and consequently $a\,(\cos\theta + \sqrt{-1}\sin\theta)$ will represent in magnitude and position a line whose length is a and inclined to the primitive at an angle θ. For we can always find whole numbers r and n such that $n \div r$ is either exactly equal or indefinitely near to $2\pi \div \theta$; consequently a_r will either be coincident or indefinitely near to coincidence with the line making $\angle\,\theta$ with the primitive.

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SECTION II.

FORMULÆ INVOLVING THE TRIGONOMETRICAL RATIOS OF TWO OR MORE ANGLES.

Formulæ for finding the sine and cosine of the sum and difference of two angles.

35. To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles.

Let the angles BAC, CAD (fig. 15) be denoted by A and B, so that $BAD = A + B$. In AD take any point P, draw PN, PQ respectively perpendicular to AB, AC; and QR, QM parallel and perpendicular to AB; then RM is a rectangle, and the angle QPR is the complement of PQR, and is therefore equal to angle RQA or A.

$$\text{Now } \sin(A+B) = \frac{PN}{AP} = \frac{NR + RP}{AP} = \frac{QM}{AP} + \frac{PR}{AP};$$

or, replacing the ratio $\frac{QM}{AP}$ by the ratios $\frac{QM}{AQ}$, $\frac{AQ}{AP}$, of which it is compounded, and similarly for $\frac{PR}{AP}$,

$$\sin(A+B) = \frac{QM}{AQ} \cdot \frac{AQ}{AP} + \frac{PR}{PQ} \cdot \frac{PQ}{AP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\text{Also, } \cos(A+B) = \frac{AN}{AP} = \frac{AM - MN}{AP} = \frac{AM}{AP} - \frac{QR}{AP};$$

or, replacing each of these ratios by two others of which it is compounded,

$$\cos(A+B) = \frac{AM}{AQ} \cdot \frac{AQ}{AP} - \frac{QR}{PQ} \cdot \frac{PQ}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

36. To express the sine and cosine of the difference of two angles in terms of the sines and cosines of the angles.

Let the angles BAC, CAD (fig. 16) be denoted by A and B, so that $BAD = A - B$. In AD take any point P, draw

PN, PQ respectively perpendicular to AB, AC; and QM, PR perpendicular and parallel to AB; then RN is a rectangle, and the angle PQR is the complement of AQM, and is therefore equal to angle QAM or A.

$$\text{Now } \sin(A-B) = \frac{PN}{AP} = \frac{QM - QR}{AP} = \frac{QM}{AP} - \frac{QR}{AP};$$

or, replacing each of these ratios by two others of which it is compounded,

$$\sin(A-B) = \frac{QM}{AQ} \cdot \frac{AQ}{AP} - \frac{QR}{QP} \cdot \frac{QP}{AP}$$
$$= \sin A \cos B - \cos A \sin B.$$

$$\text{Also, } \cos(A-B) = \frac{AN}{AP} = \frac{AM + MN}{AP} = \frac{AM}{AP} + \frac{PR}{AP};$$

or, replacing each of these ratios by two others of which it is compounded,

$$\cos(A-B) = \frac{AM}{AQ} \cdot \frac{AQ}{AP} + \frac{PR}{PQ} \cdot \frac{PQ}{AP}$$
$$= \cos A \cos B + \sin A \sin B.$$

37. The figures appear to restrict the above results to the case where A and B are positive angles, and $A + B$ less than 90°; and to require that A exceed B in the formulæ relative to $A - B$. It is true we may modify the constructions so as to be applicable to all other cases; but as these cases are numerous, the following mode of establishing the formulæ

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \;\ldots\ldots\; (1),$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B \;\ldots\ldots\; (2),$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B \;\ldots\ldots\; (3),$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B \;\ldots\ldots\; (4),$$

for angles of all magnitudes and signs, is preferable.

1. The restriction of $A > B$ may be removed from (3) and (4); for suppose $A < B$, then

$$\sin(A-B) = -\sin(B-A) = -\sin B \cos A + \cos B \sin A,$$
$$\cos(A-B) = +\cos(B-A) = \cos B \cos A + \sin B \sin A,$$

since (3) and (4) are applicable to $B - A$. Hence we have for $\sin(A-B)$ and $\cos(A-B)$, the same formulæ whether $A >$ or $< B$; therefore the above four formulæ are true in all cases where the angles A and B are positive, and $A + B < 90^\circ$, or each of them between zero and 45°.

2. Since (3) and (4) result from (1) and (2) by changing B into $-B$, it follows that (1) and (2) hold for negative values of B between 0° and -45°; and we shall now shew that they hold also for negative values of A between 0° and -45°. For suppose $C < 45^\circ$; and make $A = -C$;

$$\therefore \sin(A+B) = \sin(-C+B) = -\sin(C-B),$$
$$\cos(A+B) = \cos(-C+B) = \cos(C-B);$$

now C and B are within the limits for which (3) and (4) have been proved, and $\sin C = -\sin A$, $\cos C = \cos A$;

$$\therefore \sin(A+B) = -\{\sin C \cos B - \cos C \sin B\} = \sin A \cos B + \cos A \sin B,$$
$$\cos(A+B) = \cos C \cos B + \sin C \sin B = \cos A \cos B - \sin A \sin B,$$

which are the same as (1) and (2).

3. We shall now shew that in (1) and (2) we may extend indefinitely the positive and negative limits of A and B.

Make $A = 90^\circ + C$, C being an angle between -45° and $+45^\circ$. Then, taking the complements (Art. 12),

$$\sin(A+B) = \sin(90^\circ + C + B) = \cos(-C-B) = \cos(C+B),$$
$$\cos(A+B) = \cos(90^\circ + C + B) = \sin(-C-B) = -\sin(B+C);$$

but $\sin A = \cos(-C) = \cos C$, $\cos A = \sin(-C) = -\sin C$, and the angles B and C are within the limits;

$$\therefore \sin(A+B) = \cos C \cos B - \sin C \sin B = \sin A \cos B + \cos A \sin B,$$
$$\cos(A+B) = -\sin C \cos B - \cos C \sin B = \cos A \cos B - \sin A \sin B,$$

so that (1) and (2) still hold for values of A, the positive limit of which is 135°; and by repeating the process, it is evident that the limit may be increased by any number of right angles. Also the proof given above, that if (1) and (2) are true for positive values of $A < 45^\circ$, they are true for the same values taken negatively, may evidently be applied to the case where the positive limit of A is different from 45°. Hence (1) and (2), since they hold for all positive values of A, hold for all negative values of A. With respect to B, the same reasoning is manifestly applicable to it, and we may in the same manner indefinitely extend each of its limits. Hence the formulæ (1) and (2), and consequently (3) and (4), are demonstrated for all values of the angles A and B.

Formulæ for the multiplication of angles.

38. In the expressions for $\sin(A+B)$ and $\cos(A+B)$, suppose $B = A$, then

$$\sin 2A = 2 \sin A \cos A \text{ .... } (1),$$
$$\cos 2A = \cos^2 A - \sin^2 A \text{ .... } (2),$$

formulæ which enable us, from knowing the sine and cosine of an angle, to calculate the sine and cosine of double that angle.

If in the former we substitute successively $\sqrt{1-\sin^2 A}$ for $\cos A$, and $\sqrt{1-\cos^2 A}$ for $\sin A$, we get

$$\sin 2A = 2\sin A\sqrt{1-\sin^2 A},$$

$$\sin 2A = 2\cos A\sqrt{1-\cos^2 A},$$

where $\sin 2A$ is expressed by $\sin A$ only, and by $\cos A$ only; and in each case has two values, on account of the radical involved.

Also if in the latter we substitute successively $1-\sin^2 A$ for $\cos^2 A$, and $1-\cos^2 A$ for $\sin^2 A$, we find

$$\cos 2A = 1 - 2\sin^2 A,$$

$$\cos 2A = 2\cos^2 A - 1,$$

where $\cos 2A$ is expressed by $\sin A$ only, and by $\cos A$ only.

39. Next in the expressions for $\sin(A+B)$ and $\cos(A+B)$, suppose $B = 2A$, then

$$\sin 3A = \sin A\cos 2A + \cos A\sin 2A,$$

$$\cos 3A = \cos A\cos 2A - \sin A\sin 2A.$$

Now replace $\sin 2A$ and $\cos 2A$ by the values just found, and reduce by means of the relation $\sin^2 A + \cos^2 A = 1$, and there results

$$\sin 3A = \sin A\,(1-2\sin^2 A) + \cos A\,.\,2\sin A\cos A$$

$$= \sin A\,(1-2\sin^2 A) + 2\sin A\,(1-\sin^2 A) = 3\sin A - 4\sin^3 A,$$

$$\cos 3A = \cos A\,(2\cos^2 A - 1) - \sin A\,.\,2\sin A\cos A$$

$$= \cos A\,(2\cos^2 A - 1) - 2\cos A\,(1-\cos^2 A) = 4\cos^3 A - 3\cos A.$$

Proceeding in this manner, we may find expressions for the sines and cosines of $4A$, $5A$, &c. and of all succeeding multiples of A. There exist, however, general formulæ for the multiplication of angles, which will be given in a future Section.

The reason why $\sin mA$ when expressed by $\sin A$ has one or two values, according as m is odd or even, and $\cos mA$ when expressed by $\cos A$, has always but one value, may be thus explained. Since $n\pi + (-1)^n\alpha$ is the general value of all angles that have a given sine;

if we express $\sin mA$ in terms of $\sin A$, the result must be the value of

$$\sin m\{n\pi + (-1)^n a\} = \cos mn\pi \, . \sin(-1)^n ma = \pm \sin ma, \text{ if } m \text{ be even;}$$

$$= (-1)^n . (-1)^n \sin ma = \sin ma, \text{ if } m \text{ be odd;}$$

since the sine of any multiple of π is 0, and its cosine ± 1.

Similarly if $\cos mA$ be expressed in terms of $\cos A$, the result will be the value of $\cos m(2n\pi \pm a) = \cos 2mn\pi \, . \cos(\pm ma) = \cos ma$.

Formulæ for the division of angles.

40. Change A into $\frac{A}{2}$ in formulæ (1) and (2), Art. 38, then

$$2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A,$$

$$\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = \cos A,$$

and we have, besides, the relation $\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} = 1$.

41. First suppose $\cos A$ given, i. e. that we are required to find the sine and cosine of half an angle from knowing the cosine of the angle. Adding and subtracting the two latter of the above equations, and extracting the root, we find

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}, \quad \cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}.$$

In these expressions the radical has a double sign $\pm$; the reason why we obtain two values, equal in magnitude but of contrary signs, for each of the unknown quantities $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, is that since we are at liberty to write $2n\pi \pm a$ for A under the radical, (that being the general value of angles which have a given cosine, Art. 32,) the same substitution must also be admissible in the former members of the equations; and therefore the above formulæ must give all values comprised in

$$\sin\left(n\pi \pm \frac{a}{2}\right), \cos\left(n\pi \pm \frac{a}{2}\right), n \text{ being any integer whatever.}$$

Now, by Art. 29, if n is even, these are the same as

$$\sin\left(\pm \frac{a}{2}\right) = \pm \sin \frac{a}{2}, \quad \cos\left(\pm \frac{a}{2}\right) = \cos \frac{a}{2};$$

if n is odd, they are the same as

$$-\sin\left(\pm\frac{\alpha}{2}\right)=\mp\sin\frac{\alpha}{2}, \quad -\cos\left(\pm\frac{\alpha}{2}\right)=-\cos\frac{\alpha}{2};$$

which shew that we must have two values, equal in magnitude but of contrary signs, and no more, for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when they are determined from $\cos A$.

42. Next suppose $\sin A$ given, i. e. that we have to find the sine and cosine of half an angle from knowing the sine of the angle. Resuming the equations

$$2\sin\frac{A}{2}\cos\frac{A}{2}=\sin A,$$

$$\cos^2\frac{A}{2}+\sin^2\frac{A}{2}=1,$$

and taking the square root of their sum and difference, we have

$$\left.\begin{aligned}\cos\frac{A}{2}+\sin\frac{A}{2}&=\sqrt{1+\sin A}\\ \cos\frac{A}{2}-\sin\frac{A}{2}&=\sqrt{1-\sin A}\end{aligned}\right\}\ (1),$$

from which we easily deduce the required values

$$\sin\frac{A}{2}=\tfrac{1}{2}\sqrt{1+\sin A}-\tfrac{1}{2}\sqrt{1-\sin A},$$

$$\cos\frac{A}{2}=\tfrac{1}{2}\sqrt{1+\sin A}+\tfrac{1}{2}\sqrt{1-\sin A};$$

where we have supposed, for the present, A to be less than 90^0, and therefore $\cos\frac{1}{2}A$ to be greater than $\sin\frac{1}{2}A$; and consequently we have taken each radical with a positive sign.

Each of the expressions for $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$, on account of the two radicals which it involves, has four values; that such ought to be the case, appears from this, that they ought to give the sine and cosine of the half of all angles whose sine $=\sin A$, i. e. of all angles comprised in

$\frac{1}{2}n\pi+(-1)^n\frac{1}{2}\alpha$, n being any integer whatever, (Art. 31)

or in $\frac{1}{2}n\pi+\frac{1}{2}\alpha$, or $\frac{1}{2}(n-1)\pi+\frac{1}{2}(\pi-\alpha)$, according as n is even or odd.

Hence, taking the sines of these angles, we must have, according as n is even or odd, the values

$$\cos\frac{n\pi}{2}\sin\frac{a}{2},\quad \cos\frac{(n-1)\pi}{2}\sin\frac{\pi-a}{2},$$

$$\text{or } \pm\sin\frac{a}{2},\quad \pm\sin\frac{\pi-a}{2}\quad \text{for } \sin\frac{A}{2};$$

observing that the sine of any even multiple of $\frac{1}{2}\pi$ is 0, and its cosine ± 1, and the sine of any odd multiple of $\frac{1}{2}\pi$ is ± 1, and its cosine 0; likewise, taking the cosines of the same angles, we must have, according as n is even or odd, the values

$$\cos\frac{n\pi}{2}\cos\frac{a}{2},\qquad \cos\frac{(n-1)\pi}{2}\cos\frac{\pi-a}{2},$$

$$\text{or } \pm\cos\frac{a}{2},\qquad \pm\cos\frac{\pi-a}{2}\quad \text{for } \cos\frac{A}{2};$$

which shew that we must have four values equal two and two and of contrary signs, and no more, for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, when they are determined from $\sin A$. If a be a right angle, $\frac{\pi-a}{2}$ and $\frac{a}{2}$ will each be half a right angle and the four values are reduced to two.

§ 43. When a certain value, a, is given for $\sin A$, there are, as we have seen, four values for $\sin\frac{A}{2}$; but when of all the angles which satisfy the equation $\sin A = a$, a particular angle is assigned, there will be only single values for the sine and cosine of its half; and the appropriate ones may be ascertained by considering whether, for the particular value of A, the sine and cosine of its half are positive or negative, and which is numerically the greater; and so determining the signs which the radicals are to carry in the equations

$$\cos\frac{A}{2}+\sin\frac{A}{2} = \pm\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2} = \pm\sqrt{1-\sin A},$$

before the addition and subtraction of those equations is performed. Thus when $A < 90^0$, we have seen that both the radicals must carry a positive sign. If A be between 90^0 and 180^0, $\cos\frac{1}{2}A$ and $\sin\frac{1}{2}A$ are both positive, but the latter

is the greater, and therefore the radical in the upper equation takes a positive sign, and that in the lower equation a negative sign; which is also true when A is between 180^0 and 270^0, for although $\cos\frac{1}{2}A$ is then negative it is numerically less than the positive value of $\sin\frac{1}{2}A$. But when A lies between 270^0 and 360^0, $\sin\frac{1}{2}A$ is positive and $\cos\frac{1}{2}A$ negative, and the latter is numerically the greater; therefore both radicals must carry a negative sign.

44. Next, to trisect an angle. Changing A into $\frac{A}{3}$, the formulæ of Art. 39 give

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3},$$

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}.$$

Suppose, for instance, we have $\cos A$ given, and $\cos\frac{1}{3}A$ is required; putting $\cos A = a$, $\cos\frac{1}{3}A = z$, we have

$$z^3 - \tfrac{3}{4}z - \tfrac{1}{4}a = 0,$$

for the cubic equation to be solved in order to find $\cos\frac{1}{3}A$.

We may shew *à priori* that $\cos\frac{1}{3}A$, when determined from $\cos A = a$, must have three real values, and no more. For since all values of A which satisfy the equation $\cos A = a$, are comprised in $A = 2n\pi \pm a$, the above equation must have for roots all values of z comprised in

$$z = \cos\frac{2n\pi \pm a}{3};$$

now n is of one of the forms $3m$ or $3m \pm 1$;

$$\therefore\ z = \cos\left(2m\pi \pm \frac{a}{3}\right) = \cos\frac{a}{3},$$

$$z = \cos\left(2m\pi \pm \frac{2\pi}{3} \pm \frac{a}{3}\right) = \cos\frac{2\pi \pm a}{3};$$

so that z has three real values, and no more, unless $a = \pi$.

Formulæ relative to tangents.

45. To express the tangent of the sum or difference of two angles in terms of the tangents of the angles.

By the relation which exists between the sine, cosine, and tangent of an angle, we have

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)};$$

or, replacing $\sin(A+B)$, $\cos(A+B)$, by their values, Art. 35,

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

In order to have only tangents of A and B involved, divide numerator and denominator by $\cos A \cos B$; then

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

In the same manner, for the difference of two angles, we find

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

46. Let $B = A$, then for doubling the angle we have

$$\tan 2A = \frac{2 \tan A}{1 - \tan A}.$$

Let $B = 2A$, then

$$\tan 3A = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A};$$

and, substituting for $\tan 2A$, and reducing, we find $\tan 3A$ in terms of $\tan A$; and so on for $\tan 4A$, &c.

47. Let $A = 45^0$, then $\tan A = 1$, and

$$\tan(45^0 + B) = \frac{1 + \tan B}{1 - \tan B},$$

$$\tan(45^0 - B) = \frac{1 - \tan B}{1 + \tan B}.$$

48. To find $\tan\frac{A}{2}$ in terms of $\tan A$, change A into $\frac{A}{2}$ in the preceding formula for $\tan 2A$, and there results

$$\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \tan A,$$

which is the same as the equation of the 2nd degree,

$$\tan^2\frac{A}{2} + \frac{2}{\tan A}\cdot\tan\frac{A}{2} - 1 = 0,$$

from which we find

$$\tan\frac{A}{2} = \frac{1}{\tan A}\left(-1 \pm \sqrt{1+\tan^2 A}\right).$$

The reason why $\tan\frac{A}{2}$, when determined from $\tan A$, has two values and no more, may be assigned just as in former instances.

49. The following expressions for $\tan\frac{A}{2}$ are often met with;

$$\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}},$$

$$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}, \quad \tan\frac{A}{2} = \frac{1-\cos A}{\sin A};$$

they are easily deduced from formulæ already known; it is clear, in fact, by Arts. 40 and 41, that we have

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{1-\cos A}{1+\cos A}},$$

$$\tan\frac{A}{2} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}} = \frac{\sin A}{1+\cos A},$$

$$\tan\frac{A}{2} = \frac{2\sin^2\dfrac{A}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}} = \frac{1-\cos A}{\sin A}.$$

Certain other formulæ frequently employed.

50. The formulæ of Arts. 35, 36, expressing the sine and cosine of the sum and difference of two angles, lead to a variety of formulæ much used in Astronomy; the following are the principal ones.

Combining those formulæ by addition and subtraction, we find

$$2\sin A\cos B = \sin(A+B) + \sin(A-B),$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B),$$
$$2\cos A\cos B = \cos(A-B) + \cos(A+B),$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B),$$

which serve to resolve the product of a sine and cosine, or the product of two sines, or of two cosines, into the sum or difference of two Trigonometrical Ratios.

51. If in the preceding results we replace A by $(n-1)B$, we obtain formulæ worthy of notice as expressing the sines and cosines of multiples of an angle in terms of the sines and cosines of inferior multiples of the same angle. From the first and third we thus deduce

$$\sin nB = 2\sin(n-1)B\cos B - \sin(n-2)B,$$
$$\cos nB = 2\cos(n-1)B\cos B - \cos(n-2)B.$$

52. Again, since

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),\quad B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B),$$

expanding the sines and cosines of these equivalent values of A and B, by the formulæ of Art. 35 and 36, and combining them by addition and subtraction, we find

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$
$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$
$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$
$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

These formulæ are of great use (especially in calculations effected by logarithms) in transforming a sum or difference into a product.

53. Lastly, by division, observing in general that

$$\frac{\sin A}{\cos A} = \tan A = \frac{1}{\cot A},$$

we obtain from the preceding formulæ the following ones of scarcely less utility:

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)},$$

$$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A \pm B),$$

$$\frac{\sin A \pm \sin B}{\cos B - \cos A} = \cot \tfrac{1}{2}(A \mp B),$$

$$\frac{\cos B - \cos A}{\cos A + \cos B} = \tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B).$$

Among these formulæ the first is to be particularly remarked. It may be thus enunciated: the sum of the sines of two angles is to the difference of the sines, as the tangent of the semi-sum of the angles is to the tangent of the semi-difference.

54. We sometimes meet with transformations of Trigonometrical formulæ, of which it is not easy to perceive the origin; in such cases it is sufficient to verify them, which can never be attended with difficulty, and for that purpose we may begin with either member of the equation. For instance, to verify the relation

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B;$$

replacing $\sin(A+B)$ and $\sin(A-B)$ by their values, we have

$$\sin(A+B)\sin(A-B) = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B;$$

then, substituting $1 - \sin^2 A$ and $1 - \sin^2 B$ in the place of $\cos^2 A$, $\cos^2 B$, and reducing, we find the proposed relation.

Or, if we commence with the second member of the equation, the process will be this:

$$\sin^2 A - \sin^2 B = \tfrac{1}{2}(1 - \cos 2A) - \tfrac{1}{2}(1 - \cos 2B)$$
$$= \tfrac{1}{2}(\cos 2B - \cos 2A)$$
$$= \tfrac{1}{2} \cdot 2 \sin (A + B) \sin (A - B).$$

55. Again, if these other relations were proposed,

$$\cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}, \qquad \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}};$$

putting for $\tan \frac{A}{2}$ its value $\dfrac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$, the second members become

$$\frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}} \qquad \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}},$$

which, by (Art. 40), reduce themselves to $\cos A$ and $\sin A$, as was to be shewn.

56. The following transformations may be proposed for practice.

$$\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B,$$

$$\tan A + \cot A = 2 \operatorname{cosec} 2A,$$

$$\tan A + \sec A = \tan \left(45^0 + \frac{A}{2}\right),$$

$$\cot A + \operatorname{cosec} A = \cot \frac{A}{2},$$

$$\tan (45^0 + A) - \tan (45^0 - A) = 2 \tan 2A,$$

$$\cos A = \frac{1}{1 + \tan A \tan \frac{A}{2}},$$

$$\tan A + \tan B = \frac{\sin (A + B)}{\cos A \cos B},$$

$$\tan (A + B) - \tan A = \frac{\sin B}{\cos (A + B) \cos A}.$$

Also, supposing $A + B + C = 180^0$,

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

$$\cos A + \cos B + \cos C = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C + 1,$$

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C$$
$$+ \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C,$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

The last of these formulæ proves that we may choose, in an infinite number of ways, three numbers such that their sum shall equal their product.

57. All the preceding results have the same generality as the formulæ for the sines and cosines of $A + B$ and $A - B$, from which they have been deduced; and are therefore true for angles of any magnitude and sign.

Before entering on the succeeding Sections, the Appendix on Logarithms may be conveniently read.

SECTION III.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

Natural sines and cosines for every ten seconds of the quadrant.

58. In order that the replacing of the angles by their Trigonometrical Ratios may be attended with real utility, it is requisite that when the angle is assigned we should know the numerical values of its Trigonometrical Ratios, and conversely. The best way of attaining this object is to form Tables, in which the values of the Trigonometrical Ratios are registered side by side with the angles to which they correspond. We must therefore now shew how to calculate the sines, cosines, &c., of all angles between zero and 90^0, for every $10''$, that being the interval at which the angles succeed one another in the best tables; and it must not be thought that unnecessary accuracy is here studied, for in the present state of Astronomical Science, an error amounting to a small fraction of a second is often of serious importance. But we must first establish the truth of the following Propositions.

59. The circular measure of an angle between zero and a right angle, is greater than its sine and less than its tangent.

Let BAC, $B'AC$ (fig. 17) be two equal angles, each less than 90^0, with circular measure θ. From any point C in AC draw CB, CB' perpendiculars to AB, AB'; join BB' cutting AC in N, and with centre A and radius AB describe a circular arc cutting AC in D, and which will manifestly pass through B'. Then arc BDB' is greater than BB';

$$\therefore\ BD > BN, \text{ and } \frac{BD}{AB} > \frac{BN}{AB}, \text{ or } \theta > \sin\theta.$$

Also, admitting the principle that the boundary of a convex curvilinear figure entirely contained within another is less than that of the containing figure,

arc BDB' is less than $BC + B'C$,

$$\therefore\ BD < BC, \text{ and } \frac{BD}{AB} < \frac{BC}{AB}, \text{ or } \theta < \tan\theta.$$

60. As the angle whose circular measure is θ, is continually diminished to zero, each of the quantities $\frac{\sin\theta}{\theta}, \frac{\tan\theta}{\theta}$ continually approaches to unity, and has unity for its ultimate value.

For since θ lies between $\sin\theta$ and $\tan\theta$,

$$\frac{\theta}{\sin\theta} \text{ is nearer to unity than } \frac{\tan\theta}{\sin\theta} \text{ or } \frac{1}{\cos\theta};$$

but the ultimate value of $\frac{1}{\cos\theta}$, when θ is continually diminished, is 1; therefore, *à fortiori*, the ultimate value of $\frac{\theta}{\sin\theta}$, when θ vanishes, is unity. Also, since $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$, the ultimate value of $\frac{\tan\theta}{\theta}$, when θ vanishes, is unity.

61. If θ be the circular measure of an angle between zero and a right angle, then $\sin\theta > \theta - \frac{\theta^3}{4}$.

$$\text{For} \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$= 2\tan\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2},$$

$$> 2\cdot\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right), \text{ for } \frac{\theta}{2} < \tan\frac{\theta}{2},$$

$$\text{à fortiori} > \theta\left(1 - \frac{\theta^2}{4}\right), \text{ for } \frac{\theta}{2} > \sin\frac{\theta}{2}.$$

This result will be useful to us in estimating the degree of approximation in the next Article.

62. To find the sine of 10″.

Let θ denote the circular measure of an angle containing 10″; then since the circular measure of 180⁰ is $\pi = 3{\cdot}1415926535$, and the ratio of two magnitudes is the same whatever be the unit in which they are expressed,

$$\frac{\theta}{\pi} = \frac{10}{180 \times 60 \times 60} = \frac{1}{64800}; \quad \therefore \theta = \frac{\pi}{64800}.$$

Now $\sin 10'' > \theta - \dfrac{\theta^3}{4}, < \theta$; and on substituting for θ the above value, it is found* that these limits coincide in the first twelve places of decimals; therefore, to twelve places of decimals

$$\sin 10'' = \theta = \frac{\pi}{64800};$$

and cos 10″ is found by substituting this value in the formula $\cos 10'' = \sqrt{1 - \sin^2 10''}$.

Hence if θ be the circular measure of an angle containing n seconds, then

$$\theta = n \sin 1'';$$

for if h be the circular measure of 1″, then $\theta = nh$; but $h = \sin 1''$ exact to at least 12 places of decimals; therefore with equal exactness,

$$\theta = n \sin 1''.$$

* The calculation is as follows:

We have $\theta = {\cdot}00004\ 84813\ 68110$.

Now $\sin 10'' > \theta - \dfrac{\theta^3}{4}$,

$\therefore$ *à fortiori*, $\sin 10'' > \theta - \frac{1}{4}({\cdot}00005)^3$,

or $\sin 10'' > {\cdot}00004\ 84813\ 68110$,

$- 00000\ 00000\ 00032$,

or $\sin 10'' > {\cdot}00004\ 84813\ 68078$,

also $\sin 10'' < \theta < {\cdot}00004\ 84813\ 68110$,

therefore the value of sin 10″, correct to 12 places of decimals, is

$\sin 10'' = {\cdot}00004\ 84813\ 68$.

Substituting this value of sin 10″ in the formula $\cos 10'' = \sqrt{1 - \sin^2 10''}$, we have

$\cos 10'' = {\cdot}99999\ 99988\ 248$.

63. The sine and cosine of $10''$ being known, the sines and cosines of all angles between 0^0 and 90^0, from $10''$ to $10''$, may be computed.

Making $A = n.10''$, $B = 10''$ in the formula

$$\sin(A + B) = 2 \sin A \cos B - \sin(A - B),$$

we find

$$\sin(n + 1)\, 10'' = 2 \sin n\, 10'' . \cos 10'' - \sin(n - 1)\, 10'';$$

now $2 \cos 10''$ differs from 2 by a very small* quantity k, putting, therefore, $2 - k$ for $2 \cos 10''$, and transposing, we get

$$\sin(n + 1)\, 10'' - \sin n\, 10'' = \sin n\, 10'' - \sin(n - 1)\, 10'' - k \sin n\, 10'';$$

hence making $n = 1, 2,$ &c. $\sin 20''$, $\sin 30''$, &c. become successively known; and in general the difference of the sines of consecutive angles $(n + 1)\, 10''$ and $n\, 10''$ will be obtained by diminishing the difference of the sines of the preceding angles $n\, 10''$ and $(n - 1)\, 10''$, which is already calculated, by $k \sin n\, 10''$; so that for each new sine, the only laborious operation will be to multiply the last obtained sine by k. It is necessary to take $\sin 10''$ and $\cos 10''$ with a great many more decimal places than we mean eventually to preserve, in order that the accumulated error, in the long series of operations for forming a table of sines, may have no influence on that order of decimals which we wish to have accurate in the last results.

64. Having computed the sines of angles ascending by intervals of $10''$, from 0^0 to 60^0, the sines of angles ascending by the same interval from 60^0 to 90^0 may be found from the formula

$$\sin(60^0 + A) - \sin(60^0 - A) = 2 \cos 60^0 \sin A = \sin A,$$

$$\text{since } \cos 60^0 = \tfrac{1}{2},$$

$$\text{or } \sin(60^0 + A) = \sin A + \sin(60^0 - A),$$

by putting $A = 10''$, $20''$, &c. successively.

65. The sines of all angles from 0^0 to 90^0 being computed, no calculation is requisite for the cosines; for $\sin 10''$ is the same

* The numerical value of k is $\cdot 00000\ 00023\ 504$.

thing as $\cos 89^0.59'.50''$, $\sin 20''$ the same as $\cos 89^0.59'.40''$, and so on; so that a complete table of sines is also a complete table of cosines.

66. The tangents and secants are of course known from the sines and cosines; and complete tables of tangents and secants are also complete tables, respectively, of cotangents and cosecants, just as in the case of the sine and cosine. When the tangents of all angles as far as 45^0 are computed, the tangents of angles between 45^0 and 90^0 may be found from the formula (Art. 56)

$$\tan(45^0 + A) = \tan(45^0 - A) + 2\tan 2A,$$

by putting $A = 10''$, $20''$, &c. successively. Also, since (Art. 56)

$$\operatorname{cosec} A = \frac{1}{2}\left\{\tan\frac{A}{2} + \cot\frac{A}{2}\right\},$$

and replacing A by $90^0 + A$,

$$\sec A = \frac{1}{2}\left\{\tan\left(45^0 + \frac{A}{2}\right) + \cot\left(45^0 + \frac{A}{2}\right)\right\},$$

the tables of tangents and cotangents will give, by simple addition, the cosecants and secants of angles which are even multiples of $10''$.

67. When an angle, besides degrees and minutes, contains a number of seconds not a multiple of 10, its sine, cosine, &c. are not found exactly in the Tables; they may however be deduced from those of the angle nearest to it, as will be shewn in a future Article, on the principle that the increments of the sines, cosines, &c. of angles, are proportional to the increments of the angles; the calculations being precisely similar to those employed in treating of the logarithms of numbers. As this is a point which will be fully illustrated when we come to speak of the logarithmic sines, cosines, &c. of angles, (which are of far greater practical importance than the natural sines, cosines, &c.) it is unnecessary to say more upon it here.

68. The Tables never go beyond 45^0; for angles greater than 45^0, the sines, tangents, and secants are determined by the cosines, cotangents, and cosecants of the complements which

are less than 45^0. Thus for $\sin 60^0 . 9' . 40''$ we take out the value of $\cos 29^0 . 50' . 20''$. The arrangement of the Tables saves even the trouble of calculating these complements. Thus the pages containing the values of the Trigonometrical Ratios of angles from 29^0 to 30^0, are marked 29^0 at the top, and have on the left a descending column of minutes; at the bottom they are marked 60^0, and have on the right an ascending column of minutes; and those columns which are marked sin, tan, sec, at the top, are marked, respectively, cos, cot, cosec, at the bottom; so that here, and in general, by consulting the descriptions of the columns at the top of the page, and the descending column on the left for the minutes and seconds, we take out the sine, cosine, &c., when the angle is less than 45^0; when it is greater than 45^0, we consult the descriptions at the bottom of the page, and the ascending column on the right for the minutes and seconds.

69. Since the sines and cosines of all angles are less than 1, the sines of small angles, and the cosines of angles a little less than 90^0, are very small quantities; and when expressed decimally they will have one, two, or several cyphers between the decimal point and the first significant digit. To obviate the inconvenience of printing these cyphers, the real values of all the Trigonometrical Ratios are multiplied by 10000, so that the decimal point is moved four places to the right. Thus in the column marked N. Sine in the Table for 29^0, we find the sines, or Natural Sines as they are called, printed with four places of figures before the decimal point. To deduce the real value of any Trigonometrical Ratio of an angle from its tabular value, we must remove the decimal point four places to the left.

Direct calculation of the sines and cosines of certain angles for the Verification of the Tables.

70. As an error which at first affects only the decimal places of a high order, may, in the long series of operations necessary for computing a Table of sines, at length affect the decimal places of a much lower order, and therefore produce a considerable error in the last results; and as any error of calculation will be transmitted through all the succeeding re-

sults, we perceive that it is impossible to preserve the same number of decimal places exact to the end, which we prove to be exact in the value of sin 10″, at the outset. In order therefore to check any error of calculation, and to ascertain the degree of precision upon which we may reckon in the Tables, we shall now shew how to find expressions for several sines and cosines from which we can obtain absolute approximations to their values; then the decimals which are common to these values, and to the tabular values of the same sines and cosines furnished by the above successive calculations, will indicate with certainty the decimals which we may look upon as exact in the latter, and in the intermediate tabulated results. These verifications may be best obtained by calculating the sines and cosines of all angles from 0^0 to 90^0, at intervals of 9^0.

71. To find the value of sin 18^0.

Let A denote 18^0, then since

$$\sin 36^0 = \cos (90^0 - 36^0) = \cos 54^0,$$

$$\sin 2A = \cos 3A,$$

$$\begin{aligned} \text{or}\quad 2 \sin A \cos A &= \cos 2A \cos A - \sin 2A \sin A \\ &= (1 - 2\sin^2 A)\cos A - 2\sin^2 A \cos A, \end{aligned}$$

and dividing by $\cos A$, we get

$$4 \sin^2 A + 2 \sin A = 1;$$

therefore, solving the equation,

$$\sin A = \frac{\pm\sqrt{5} - 1}{4},$$

and since sin 18^0 is a positive quantity,

$$\sin 18^0 = \cos 72^0 = \tfrac{1}{4}(\sqrt{5} - 1).$$

72. With this value of sin 18^0 we find successively

$$\cos^2 18^0 = 1 - \sin^2 18^0 = 1 - \frac{6 - 2\sqrt{5}}{16}$$

$$\text{or}\quad \cos 18^0 = \sin 72^0 = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}};$$

$$\cos 36^0 = 1 - 2\sin^2 18^0 = 1 - 2\,\frac{6 - 2\sqrt{5}}{16}$$

$$\text{or } \cos 36^0 = \sin 54^0 = \tfrac{1}{4}(\sqrt{5} + 1),$$

$$\sin 36^0 = \cos 54^0 = \sqrt{1 - \cos^2 36^0} = \tfrac{1}{4}\sqrt{10 - 2\sqrt{5}}.$$

From the value of sin 54^0, we see that the negative sign in Art. 71 gives us sin (-54); and if $A = -54^0$, $\sin 2A = \cos 3A$.

73. Also, substituting successively the values of sin 18^0, sin 54^0, in the formulæ which give $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of sin A, Art. 42, we get

$$\sin 9^0 = \tfrac{1}{4}\sqrt{3 + \sqrt{5}} - \tfrac{1}{4}\sqrt{5 - \sqrt{5}},$$

$$\cos 9^0 = \tfrac{1}{4}\sqrt{3 + \sqrt{5}} + \tfrac{1}{4}\sqrt{5 - \sqrt{5}},$$

$$\sin 27^0 = \tfrac{1}{4}\sqrt{5 + \sqrt{5}} - \tfrac{1}{4}\sqrt{3 - \sqrt{5}},$$

$$\cos 27^0 = \tfrac{1}{4}\sqrt{5 + \sqrt{5}} + \tfrac{1}{4}\sqrt{3 - \sqrt{5}}.$$

Hence, recalling the value of $\sin 45^0 = \frac{1}{2}\sqrt{2}$, we have the sines and cosines of 0^0, 9^0, 18^0, 27^0, 36^0, 45^0; that is, the sines and cosines of all angles from 0^0 to 90^0 at intervals of 9^0. And as their expressions are sufficiently simple, and contain only square roots, we may obtain their values with as many exact decimals as we please, and employ them to verify the tables of sines and cosines.

74. On account of the simplicity of the expressions for sin 18^0 and sin 54^0, the following formulæ may be constructed with them, which are peculiarly fitted to verify the tables, as they require only the operations of addition and subtraction.

$$\sin(36^0 + A) - \sin(36^0 - A) = 2\cos 36^0 \sin A = \frac{1 + \sqrt{5}}{2}\sin A,$$

$$\sin(72^0 + A) - \sin(72^0 - A) = 2\cos 72^0 \sin A = \frac{-1 + \sqrt{5}}{2}\sin A,$$

therefore, subtracting and transposing, we get

$$\sin A + \sin(72^0 + A) - \sin(72^0 - A) = \sin(36^0 + A) - \sin(36^0 - A).$$

Again,

$$\cos(36^0 + A) + \cos(36^0 - A) = 2\cos 36^0 \cos A = \frac{1+\sqrt{5}}{2}\cos A,$$

$$\cos(72^0 + A) + \cos(72^0 - A) = 2\cos 72^0 \cos A = \frac{-1+\sqrt{5}}{2}\cos A.$$

therefore, subtracting and transposing,

$$\cos A + \cos(72^0 + A) + \cos(72^0 - A) = \cos(36^0 + A) \\ + \cos(36^0 - A).$$

Giving any value to A in these formulæ, we get a relation between the sines or cosines of a certain number of angles; which will be satisfied by the tabular values of these sines and cosines, if the tabular values be correct.

Logarithmic Tables of sines, cosines, &c.

75. In practice it is far more useful to have the logarithms of the numbers which express the Trigonometrical Ratios, than the numbers themselves. As the sines and cosines of all angles, and the tangents of angles less than 45^0, are less than unity, their logarithms are negative; and in order to avoid the introduction and use of negative quantities, the logarithms of the Trigonometrical Ratios are all increased by the addition of the number 10, and are so registered in the tables of log-sines, log-cosines, &c. In the adaptation of formulæ involving sines, cosines, &c. to logarithmic calculation, the tabular or augmented logarithms, which we shall denote by the symbol L, are always understood; so that when we have obtained a logarithmic equation expressed by real logarithms, we cannot proceed to calculate from it by the aid of the tables, until we have first replaced log-sin A, log-cos A, &c. by $(L \sin A - 10)$, $(L \cos A - 10)$, &c.

76. In making use of the Tables of log-sines, log-cosines, &c. for purposes of actual calculation, the two main problems which arise are (1) Any angle being assigned, to find by means of the tables, its log-sine, log-cosine, &c.; and (2) A log-sine, log-cosine, &c. being given, to find the corresponding angles.

If the proposed angle consist of degrees, minutes, and a multiple of 10″, then its log-sin, log-cos, &c. may be taken directly out of the tables; but if the number of seconds be not a multiple of 10, we must have recourse to the principle of proportional parts, and must make calculations precisely similar to those indicated in treating of the logarithms of numbers. This amounts to considering the differences of the log-sines, log-cosines, &c. of any angles, as proportional to the differences of the angles; and this proportion, though inexact, gives in general a sufficient approximation.

77. The following Examples will sufficiently illustrate the mode of proceeding in the first Problem, which is:

To find the log-sine, log-cosine, &c. of an angle not exactly given in the tables.

Ex. 1. To find log-sin $6^0.32'.37'',8$.

The tables give for the angles between which the proposed one lies,

$$\log \sin 6^0.32'.30'' = 9{\cdot}0566218,$$

$$\log \sin 6^0.32'.40'' = 9{\cdot}0568054,$$

the difference of which is 1836 (the last figure being a decimal of the seventh order); if therefore x be the quantity to be added to the former logarithm to give log-sin $6^0.32'.37'',8$, we have the proportion

$$10 : 7{\cdot}8 :: 1836 : x,$$

$$\therefore\ x = 183{\cdot}6 \times 7{\cdot}8 = 183{\cdot}6 \times 7 + 183{\cdot}6 \times {\cdot}8$$

$$= 1285{\cdot}2 + 146{\cdot}88 = 1432{\cdot}08,$$

$$\therefore\ \log \sin 6^0.32'.37'',8 = 9{\cdot}0566218 + {\cdot}000143208$$

$$= 9{\cdot}0567650.$$

The calculation may be conveniently arranged thus:

log sin $6^0.32'.30''$	(diff. for 10″ 1836)		=	9·0566218
for	7″	(183·6 × 7)	=	12852
for	0″,8	(18·36 × 8)	=	14688
∴ log sin $6^0.32'.37'',8$			=	9·0567650.

And we see that here, and in all other cases, as far as tens of seconds, we take the log. directly out of the Tables; for the units of seconds we add the product of the given difference for 10″ by those units, setting it one place to the right; for the tenths of a second we add the product of the diff. for 10″ by those tenths, setting it two places to the right; and so on.

Ex. 2. To find log-cos $83^0.27'.22'',2$.

log-cos $83^0.27'.30''$ (diff. for $10''$ 1836)		=	9·0566218
for	$-7''$	=	12852
for	$-0'',8$	=	14688
∴ log-cos $83^0.27'.22'',2$			9·0567650.

In this case, since the cosine and therefore log-cos is increased as the angle is diminished, we take from the tables the angle which is next *greater* than the proposed angle, and subtract the seconds from the angle, whilst we add the difference to its log-cosine. The same course must be pursued for log-cot and log-cosec.

Ex. 3. To find log-tan $8^0.13'.52'',76$.

log-tan $8^0.13'.50''$ (diff. for $10''$ 1486)		=	9·1603083
for	$2''$	=	2972
for	$0'',7$	=	10402
for	$0'',06$	=	8916
∴ log-tan $8^0.13'.52'',76$		=	9·1603493.

Ex. 4. To find log-cot $81^0.46'.7'',24$.

log-cot $81^0.46'.10''$ (diff. for $10''$ 1486)		=	9·1603083
for	$-2''$	=	2972
for	$-0'',7$	=	10402
for	$-0'',06$	=	8916
∴ log-cot $81^0.46'.7'',24$			9·1603493.

78. We now come to the second Problem, which is: Having given a log-sin, log-cos, &c. to determine the angle.

Ex. 1. To find the angle whose log-sin is 9·0567650.

In the tables the log-sin immediately inferior is

$$\text{log-sin } 6^0.32'.30'' = 9{\cdot}0566218,$$

the difference between which and the given logarithm is 1432, and the tabular difference for 10″, that is, the difference between log-sin 6°. 32′. 30″ and log-sin 6°. 32′. 40″, is 1836; hence, if n be the number of seconds to be added to 6°. 32′. 30″ we have the proportion

$$n : 10 :: 1432 : 1836,$$

$$\therefore n = \frac{14320}{1836} = 7 + \frac{1468}{1836} = 7{\cdot}8,$$

$$\therefore \text{ the required angle is } 6^0.32'.37'',8.$$

The calculation may be conveniently arranged thus:

log-sin A =	9·0567650		
for	9·0566218	(diff. for 10″ 1836)	6°. 32′. 30″
1st rem.	1432		
for	12852	....	7″
2nd rem.	1468		
for	14688	....	0″,8

$$\therefore \text{ the angle } A = 6^0.32'.37'',8$$

And we see that here, and in all other cases, taking the nearest value in the Tables, we get the angle as far as tens of seconds, and, subtracting, a first remainder; next taking that multiple of the difference for 10″ which when set one place to the right is next less than the first remainder, we get the units of seconds, and, subtracting, a second remainder; similarly, taking that multiple of the difference for 10″ which when set one place to the right is next less than the second remainder, we get the tenths of seconds, and so on.

Ex. 2. To find the angle whose log-cos is 9·0567650.

log-cos A = 9·0567650

for	9·0568054 (diff. for 10″ 1836)		$83^\circ.\ 27'.\ 20''$
1st rem.	404		
for	3672	....	$2''$
2nd rem.	368		
for	3672	....	$0'',2$

∴ the angle $A = 83^\circ.\ 27'.\ 22'',2$

Ex. 3. To find the angle whose log-tan is 9·1603493.

log-tan A = 9·1603493

for	9·1603083 (diff. for 10″ 1486)		$8^\circ.\ 13'.\ 50''$
1st rem.	410		
for	2972	....	$2''$
2nd rem.	1128		
for	10402	....	$0'',7$
3rd rem.	878		
for	8916	....	$0'',06$

∴ the angle $A \doteq 8^\circ.\ 13'.\ 52'',76$

Ex. 4. To find the angle whose log-cot is 9·1603493.

log-cot A = 9·1603493

for	9·1604569 (diff. for 10″ 1486)		$81^\circ.\ 46'.\ 0''$
1st rem.	1076		
for	10402	....	$7''$
2nd rem.	358		
for	2972	....	$0'',2$
3rd rem.	608		
for	5944	....	$0'',04$

∴ the angle $A = 81^\circ.\ 46'.\ 7'',24$

As examples of finding the logarithms when the angle is given, and *vice versa*, the following results may be verified.

$$A = 59^\circ.\ 37'.\ 42'',18,\quad B = 59^\circ.\ 37'.\ 40''.$$

	L Sin	L Cos	L Tan	L Cot
A	9·9358921	9·7038130	10·2320795	9·7679205
B	9·9358894	9·7038204	10·2320690	9·7679310
	Diff. for 10″, 124	Diff. for 10″, 359	Diff. for 10″, 483	Diff. for 10″, 483

79. It may be worth while to give here an exact investigation of the above rules for finding the values of the sine, cosine, log-sine, log-cos, &c. of an angle not exactly given in the tables; especially as it will shew clearly (which is a point of considerable importance) the cases to which those rules are inapplicable.

We have, by the formulæ of Arts. 49 and 56,

$$\begin{aligned}\sin(\theta+h)-\sin\theta &= \sin\theta\cos h+\cos\theta\sin h-\sin\theta\\ &= \cos\theta\sin h-\sin\theta(1-\cos h)\\ &= \cos\theta\sin h\left(1-\tan\theta\tan\frac{h}{2}\right)\qquad(1),\end{aligned}$$

$$\begin{aligned}\cos(\theta+h)-\cos\theta &= \sin\theta\sin h-\cos\theta(1-\cos h)\\ &= \sin\theta\sin h\left(1-\cot\theta\tan\frac{h}{2}\right)\qquad(2),\end{aligned}$$

$$\begin{aligned}\tan(\theta+h)-\tan\theta &= \frac{\sin h}{\cos(\theta+h)\cos\theta}=\frac{\sin h}{\cos^2\theta\cos h(1-\tan\theta\tan h)}\\ &= \frac{\sec^2\theta\tan h}{1-\tan\theta\tan h}\qquad(3).\end{aligned}$$

Now suppose h to be the circular measure of a small angle of about 10″, then, as has been proved, to about twelve places of decimals, $\sin h$ and $\tan h$ are both equal to h, which is a small quantity less than ·00005; therefore, provided $\tan\theta$ be not very large in (1) and (3), nor $\cot\theta$ very large in (2), $\tan\theta\tan\dfrac{h}{2}$ may be neglected in (1) in comparison with unity, and similarly for the others; and the above results become, in a state sufficiently accurate for the Tables which it is necessary to use,

$$\begin{aligned}\sin(\theta+h)-\sin\theta &= h\cos\theta,\\ \cos(\theta-h)-\cos\theta &= -h\sin\theta,\\ \tan(\theta+h)-\tan\theta &= h\sec^2\theta.\end{aligned}$$

Similarly, it may be shewn that

$$\sec(\theta + h) - \sec\theta = h \tan\theta \sec\theta,$$

$$\cot(\theta - h) - \cot\theta = h \operatorname{cosec}^2\theta,$$

$$\operatorname{cosec}(\theta - h) - \operatorname{cosec}\theta = h \cot\theta \operatorname{cosec}\theta.$$

80. Hence if we denote by $f(\theta)$ any one of the quantities $\sin\theta$, $\cos\theta$, &c. we have

$$f(\theta \pm h) - f(\theta) = \pm Ph \ldots\ldots (4)$$

(taking the upper or lower sign, according as the Trigonometrical Ratio denoted by $f(\theta)$ increases in the first quadrant with the angle, or the contrary) where P depends only on θ and does not involve h; a result which shews that all the Trigonometrical Ratios alter at a uniform rate with the angle, provided the whole alteration of the angle be small, and the angle itself not in the excepted cases, i. e. not near 90^0 for the sine and tangent, and not near zero or 180^0 for the cosine and cotangent.

81. Now suppose k to be the circular measure of $10''$, then

$$f(\theta \pm k) - f(\theta) = Pk;$$

but $f(\theta \pm k) - f(\theta)$ is immediately known from the tables, if they be for every $10''$, and is the quantity registered in the column marked Diff., let it be denoted by D. Also let the value of the angle θ in degrees, &c. be A, and the value of h in seconds be n, then $P = \frac{D}{k}$, and $\frac{h}{k} = \frac{n}{10}$; therefore, substituting in (4), we get

$$f(A \pm n) = f(A) + \frac{n}{10} D,$$

which expresses that if A be an angle given in the tables, then to find the sine, cosine, &c. of $A \pm n$, we must increase $\sin A$, $\cos A$, &c. by a quantity which is to the difference for $10''$ as n to 10.

82. From this, the solution of the converse Problem immediately follows; for we have

$$n = \frac{10}{D} \{f(A \pm n) - f(A)\},$$

which expresses that to get the angle corresponding to a given value $f(A \pm n)$, lying between the two tabular values $f(A)$ and $f(A + 10'')$, we must increase or diminish A by a number of seconds which is to 10 as the excess of the given value above $f(A)$ is to the tabular difference for $10''$.

83. Exactly in the same manner we must proceed with the log-sin, log-cos, &c.

For, $f(\theta \pm h) = f(\theta) + Ph$ gives, μ being the modulus of the common system,

$$\log f(\theta \pm h) = \log f(\theta) + \log\left(1 + \frac{Ph}{f(\theta)}\right)$$

$$= \log f(\theta) + \frac{\mu Ph}{f(\theta)}, \text{ very nearly (Art. 28, App.)} \ldots (5)$$

$$\therefore \log f(\theta \pm k) = \log f(\theta) + \frac{\mu Pk}{f(\theta)}.$$

But $\log f(\theta \pm k) - \log f(\theta)$ is the difference given in the tables, $= \delta$ suppose, so that $\frac{\mu P}{f(\theta)} = \frac{\delta}{k}$. Hence, substituting in (5), and replacing θ and h by their measures in degrees, and in seconds, respectively; and remembering that $\frac{h}{k} = \frac{n}{10}$, we get

$$\log f(A \pm n) = \log f(A) + \frac{n}{10}\delta,$$

$$\text{and } n = \frac{10}{\delta}\{\log f(A \pm n) - \log f(A)\},$$

which are nothing more than the algebraical expressions for the Rules followed in the examples of Arts. 77, 78.

For the particular case of log-sin, the steps would be

$$\sin(\theta + h) = \sin\theta + \cos\theta \, . \, h,$$

$$\therefore \text{log-sin}(\theta + h) = \text{log-sin}\,\theta + \log(1 + \cot\theta \, . \, h),$$

$$\text{or log-sin}(\theta + k) - \text{log-sin}\,\theta = \mu \cot\theta \, . \, h.$$

Now suppose $h = k$, then $\delta = \mu \cot\theta \, . \, k$, or $\mu \cot\theta = \delta \div k$;

$$\therefore \text{log-sin}(A + n) - \text{log-sin}\,A = \frac{\delta}{k}h = \frac{n}{10}\delta.$$

84. It may be observed that the same differences are common to log-tan and log-cot. For

$$\tan A = \frac{1}{\cot A},$$

$$\therefore \text{log-tan}\,A = -\text{log-cot}\,A,$$

$$\text{log-tan}(A + 10'') = -\text{log-cot}(A + 10'');$$

$$\therefore \text{log-tan}(A + 10'') - \text{log-tan}\,A = -\{\text{log-cot}(A + 10'') - \text{log-cot}\,A\}.$$

Hence the column of differences is printed between the columns of log-tans and log-cots, and serves as a column of increments to the former, and of decrements to the latter, corresponding to an increment of $10''$ in the angle. The same property may be proved exactly in the same way for the logarithms of any other pair of the Trigonometrical Ratios which are reciprocals of one another.

85. The expression for $\sin(\theta+h)-\sin\theta$ shews that when θ is nearly a right angle, its value, and consequently the value of log-sin $(\theta+h)$ − log-sin θ, is exceedingly small; and similarly the expression for $\cos(\theta-h)-\cos\theta$ shews that its value, as well as that of log-cos $(\theta-h)$ − log-cos θ, is exceedingly small when θ is nearly zero or two right angles. Now the degree of accuracy with which we can find the angle corresponding to a given value of log-sin A, or log-cos A, lying between two tabulated values, depends upon the magnitude of their difference δ. Hence, when an angle to be found is small or nearly equal to two right angles, we must avoid determining it by its cosine; when it is nearly a right angle, we must avoid determining it by its sine; the reason in both cases being that, for angles of those magnitudes, the successive logarithmic numbers are too nearly alike to enable us, with the help of tables to only seven places, to obtain a precise value of the angle. In fact the cosine of a small angle changes so slowly, that it is only when the angle exceeds $2\frac{1}{2}^\circ$, that the addition of $1''$ to the angle, alters log-cos by a decimal unit of the seventh order; and the same is of course true for the log-sin of all angles above $87\frac{1}{2}^\circ$.

Hence, when an angle A is to be found from equations of the form
$$\sin A = a,\quad \cos A = a,$$
where a is nearly equal to 1, we may replace them by the equations
$$\sin\left(45^\circ-\frac{A}{2}\right)=\sqrt{\frac{1-\cos(90^\circ-A)}{2}}=\sqrt{\frac{1-\sin A}{2}}=\sqrt{\frac{1-a}{2}},$$
$$\sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}=\sqrt{\frac{1-a}{2}},$$
which are free from the inconvenience attending the original formulæ.

86. Since when an angle A is less than 45°, the variation of its sine produced by a given variation in the angle, which is proportional to $\cos A$, is greater than the variation of the cosine produced by the same variation of the angle, which is proportional to $\sin A$; and *vice versâ* when the angle is greater than 45°; we may expect to obtain an angle less than 45° with greater numerical precision when determined

by its sine than when determined by its cosine; and an angle greater than 45°, with greater precision when determined by its cosine, than when determined by its sine.

87. Again, the expression for $\tan(\theta+h)-\tan\theta$ shews that when θ is nearly a right angle, its value is very large, and far from increasing in the same proportion as h, because the term $\tan\theta\tan h$ in its denominator cannot be neglected. Hence, when an angle to be found is nearly equal to a right angle, we must avoid determining it by its tangent; because for angles of that magnitude the variations in the successive logarithmic numbers are far too irregular to allow us to employ the common principle of the increment of log-tan being proportional to the increment of the angle. In fact we find in the tables

$$\begin{aligned} &L\tan 89^\circ.58' &&= 13{\cdot}2352438 && \text{diff. } 377886, \\ &L\tan 89^\circ.58'.10'' &&= 13{\cdot}2730324 && \text{diff. } 413927, \\ &L\tan 89^\circ.58'.20'' &&= 13{\cdot}3144251 && \end{aligned}$$

where the differences are not nearly equal.

Hence, when an angle A is to be determined from an equation of the form $\tan A = a$, where a is considerable, so that A is nearly 90^0, we must replace it by the formula

$$\tan(A-45^0)=\frac{\tan A-1}{\tan A+1}=\frac{a-1}{a+1};$$

for the angle determined is $A-45^\circ$, which does not differ much from 45°; and for that value of the angle, the increment of the log-tan is nearly proportional to the increment of the angle.

88. Supposing h so small that we may put $\sin h=h$, $\tan\frac{1}{2}h=\frac{1}{2}h$, we have from Art. 79

$$\frac{\sin(\theta+h)}{\sin\theta}=1+(h\cot\theta-\tfrac{1}{2}h^2);$$

$$\therefore \log\frac{\sin(\theta+h)}{\sin\theta}=\mu(h\cot\theta-\tfrac{1}{2}h^2-\tfrac{1}{2}h^2\cot^2\theta),$$

$$\text{or } \text{log-sin}(\theta+h)-\text{log-sin}\,\theta=\mu h\cot\theta-\tfrac{1}{2}\mu h^2\operatorname{cosec}^2\theta.$$

If therefore θ be very small, and consequently $\operatorname{cosec}\theta$ very large, in order that the second term may be neglected h must be exceedingly small. On this account it is necessary, for the first five degrees of the quadrant, to have the Tables of log-sin, &c. computed to every second. With the ordinary Tables we cannot determine a very small angle from its log-sine, and *vice versâ;* because in those Tables the principle that the increment of log-sin is proportional to the increment of the angle, does not hold.

Precisely the same observations hold with respect to the log cos of an angle nearly equal to a right angle. Also, since $\log \tan \theta = \log \sin \theta - \log \cos \theta$, we see that log tan θ will be affected, and so log cot θ. On the whole then we have the following list of values near to which if an angle be situated, the principle of proportional parts becomes inapplicable to determine it by means of the Trigonometrical Ratio set opposite; in the first column, on account of the irregularity of the increments, and in the second, on account of their being insensible.

sin		$\frac{1}{2}\pi$	log sin	0	$\frac{1}{2}\pi$
cos		0	log cos	$\frac{1}{2}\pi$	0
tan	$\frac{1}{2}\pi$		log tan	0, $\frac{1}{2}\pi$	
cot	0		log cot	0, $\frac{1}{2}\pi$	
sec	$\frac{1}{2}\pi$		log sec	$\frac{1}{2}\pi$	0
cosec	0		log cosec	0	$\frac{1}{2}\pi$

SECTION IV.

SOLUTION OF TRIANGLES.

Relations among the sides and angles of a triangle.

89. In what follows, we shall denote the angles of triangles by the letters A, B, C, placed at the angular points, and the sides respectively opposite to them by a, b, c. If the triangle be right-angled, then C shall be the right angle, and c the hypothenuse. We shall first demonstrate the fundamental Propositions on which the solution of rectilinear triangles depends.

In a right-angled triangle, the side opposite to an acute angle is equal to the hypothenuse multiplied by the sine of the angle; and the side adjacent to an acute angle is equal to the hypothenuse multiplied by the cosine of the angle.

For if ABC (fig. 5) be a right-angled triangle, and A one of its acute angles to which the side a is opposite and the side b adjacent,

$$\sin A = \frac{a}{c}, \quad \therefore a = c \sin A,$$

$$\cos A = \frac{b}{c}, \quad \therefore b = c \cos A.$$

90. In a right-angled triangle, the side opposite to an acute angle is equal to the other side multiplied by the tangent of the angle; and the side adjacent to an acute angle is equal to the other side multiplied by the cotangent of the angle.

For $$\tan A = \frac{a}{b}, \quad \therefore a = b \tan A,$$

$$\cot A = \frac{b}{a}, \quad \therefore b = a \cot A.$$

91. In any triangle, the sines of the angles are to one another as the sides opposite.

Let A and B be any two angles of the triangle ABC (fig. 18), and as one of them is necessarily acute, let it be B; and

according as the other angle A is acute or obtuse, draw CD perpendicular to BA, or BA produced.

If the perpendicular fall within the triangle, the two right-angled triangles ACD, BCD, give

$$CD = b \sin A, \qquad CD = a \sin B,$$

$$\therefore\ b \sin A = a \sin B, \qquad \text{or} \quad \frac{\sin A}{\sin B} = \frac{a}{b}.$$

If the perpendicular fall upon the side BA produced, the triangle ACD gives $CD = b \sin CAD = b \sin CAB = b \sin A$, because CAD is the supplement of CAB; and the triangle BCD gives, as before, $CD = a \sin B$; therefore, in this case also,

$$\frac{\sin A}{\sin B} = \frac{a}{b}.$$

If $A = 90^0$, we still have, in conformity with the theorem,

$$\sin B = \frac{b}{a}.$$

92. In any triangle, the square of any side is equal to the sum of the squares of the two other sides, diminished by twice the product of these sides and the cosine of the included angle; that is,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Let ABC (fig. 18) be any triangle, and as one of the angles A, B, must be acute, let it be B, and according as the angle A is acute or obtuse draw CD perpendicular to BA, or BA produced.

When the angle A is acute, we have (Euclid, II. 13.)

$$BC^2 = AC^2 + AB^2 - 2AB \times AD,$$

$$\text{or}\ a^2 = b^2 + c^2 - 2c \times AD,$$

but from the right-angled triangle ADC, $AD = b \cos A$,

$$\therefore\ a^2 = b^2 + c^2 - 2bc \cos A.$$

When the angle A is obtuse, we have (Euclid, II. 12.)

$$a^2 = b^2 + c^2 + 2AB \times AD,$$

but the triangle CAD gives

$$AD = b \cos CAD = -b \cos CAB = -b \cos A,$$

because CAD is the supplement of CAB; therefore, in this case also,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

If $A = 90^0$, we still have, in conformity with the theorem,

$$a^2 = b^2 + c^2.$$

93. Hence, whether A be an acute, or the obtuse angle of a triangle, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

or, the cosine of any angle of a triangle is equal to the fraction whose numerator is the sum of the squares of the containing sides diminished by the square of the opposite side, and denominator twice the product of the containing sides.

94. As was before stated, in a triangle there are six parts; and if values for any three be given (one of them being a side), and it be possible for a triangle to be constructed with such values, the other parts may be determined. The cases of impossibility are (1) when two angles are given having their sum greater than 180^0; (2) when three sides are given of which one is not less than the sum of the other two; and (3) when two sides and an angle opposite to one of them are given, and the sine of the angle is greater than the ratio which the side to which it is opposite bears to the other given side. The two first cases are evident from Euclid. For the third, make at the point A in the indefinite straight line AB (fig. 19) $\angle BAC$ equal to the given angle, and AC equal the given adjacent side, and draw CN perpendicular to AB; then $CN = AC \sin A$, which must be not greater than the other given side, otherwise a circle described from centre C with radius CB equal to that side, will not meet AB, and the triangle will be impossible; hence $AC \sin A$ must not be greater than CB.

If the three angles only of a triangle are given, we cannot determine it in magnitude, but we may in species; for the theorem of Art. 91 will enable us to determine the several ratios of its sides to one another.

95. If by the theorem of Art. 92 we form expressions for cos B and cos C similar to that for cos A, we shall have three equations, by

means of which we may determine three of the six parts of a triangle when the other three are known; excluding, of course, the cases where the triangle is impossible, or the three angles only are given. Similarly, from the theorem of Art. 91, which we shall see hereafter to be a consequence of the one just mentioned, we get the equations

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \frac{\sin C}{\sin B} = \frac{c}{b},$$

by means of which, together with the relation $A + B + C = 180^\circ$, we may determine three parts of a triangle when the other three are given.

As however the above formula for $\cos A$ is not convenient for logarithmic computation, (for to be convenient for logarithmic computation, a formula must be composed of factors which require only addition and subtraction for their computation preparatory to the application of logarithms,) we shall, in the next place, proceed to transform these results so as to be convenient for the application of logarithms; and to give particular modes of solution adapted to the several particular cases.

Solution of right-angled triangles.

96. In order to solve the triangle, two other parts in addition to the right angle must be given, one of them being a side; hence there will be four cases, the data in them being, respectively, the hypothenuse and an angle; a side and an angle; the hypothenuse and a side; and the two sides.

97. (1) Having given the hypothenuse c, and an angle A, to find the other angle B, and the sides a and b (fig. 5).

First, we have $B = 90^\circ - A$; also

$$\frac{a}{c} = \sin A, \qquad \frac{b}{c} = \cos A;$$

$$\therefore \; a = c \sin A, \qquad b = c \cos A;$$

or, in logarithms,

$$\log a = \log c + L \sin A - 10, \qquad \log b = \log c + L \cos A - 10;$$

where, by $L \sin A$, $L \cos A$, are meant the augmented logarithms as furnished by the tables (Art. 75), exceeding the real logarithms by 10; and which, as before stated, are invariably to be used in formulæ prepared for calculation by means of the tables.

98. (2) Having given a side a, and the angle opposite to it A, to find the other angle B, the side b, and the hypothenuse c.

First, we have $B = 90^0 - A$; also

$$\frac{a}{c} = \sin A, \qquad \frac{a}{b} = \tan A,$$

$$\therefore\ c = \frac{a}{\sin A}, \qquad b = a \cot A,$$

which may be put into logarithms, as above.

If the tables contain L cosec, we must determine c from the equation $c = a \operatorname{cosec} A$, as it requires only the addition of logarithms.

99. (3) Having given the hypothenuse c, and a side a, to find the remaining side b, and the two angles.

First, we have

$$b^2 = c^2 - a^2,$$

$$\therefore\ b = \sqrt{(c+a)(c-a)},$$

a form adapted to logarithmic computation; also

$$\sin A = \frac{a}{c}, \qquad \text{and } B = 90^0 - A.$$

If we begin by finding the angles, we may obtain b by the formula $b = c \cos A$.

100. (4) Having given the two sides a and b, to find the hypothenuse c, and the angles.

First, we have

$$\tan A = \frac{}{b}, \qquad \text{and } B = 90^0 - A;$$

then c is obtained by the relation

$$c = \frac{}{\sin A}, \qquad \text{or } c = a \operatorname{cosec} A.$$

We may also find c directly by the relation $c = \sqrt{a^2 + b^2}$; but as $a^2 + b^2$ cannot be resolved into factors, this formula is not suited to logarithmic computation, and it is better to obtain c from the value of A previously found.

101. In all the above cases, when the angle sought is either very small, or nearly a right angle, and is to be determined by a formula which for that particular value of the angle is incompetent to furnish great numerical accuracy, the formula must be transformed into another free from that defect, in the mode explained at Arts. 85—87.

The above methods may be verified upon the sides and angles of the following triangle:

$$a = 1540{\cdot}374 \qquad \log a = 3{\cdot}1876262,$$
$$b = 902{\cdot}708 \qquad \log b = 2{\cdot}9555475,$$
$$c \;\; 1785{\cdot}395 \qquad \log c = 3{\cdot}2517343,$$

	L sin	L cos	L tan
$A = 59^\circ.37'\,42''$	9·9358919	9·7038132	10·2320786.

Solution of oblique-angled triangles.

102. Since we must have three parts given, one of which is a side, there will be only four distinct cases, the data in them being, respectively, two angles and a side; two sides and an angle opposite to one of them; two sides and the included angle; and three sides.

103. (1) Having given a side a, and two angles, to find the other parts (fig. 18).

Subtracting the sum of the two known angles from 180°, we find the third angle. We then find b and c from the equations

$$b = a\frac{\sin B}{\sin A}, \qquad c = a\frac{\sin C}{\sin A};$$

or, in logarithms,

$\log b = \log a + L\sin B - L\sin A$, $\log c = \log a + L\sin C - L\sin A$,

the tens destroying one another.

104. (2) Having given two sides a, b, and the angle A opposite to one of them, to find the third side c and the two other angles.

We must find the angle B from the relation $\sin B = \dfrac{b}{a}\sin A$; then

$$C = 180^\circ - (A + B), \text{ and } c = a\frac{\sin C}{\sin A}.$$

105. In this case, whenever the given angle is acute, and the side opposite to it is less than the side adjacent to it, there will be two triangles which have the data of the Problem.

At the point A (fig. 19) in the indefinite straight line AB, make the angle BAC equal to the given acute angle A, and AC equal to the greater of the given sides b; with centre C and radius CD equal to the other given side a, let a circle be described; then it must meet AB, otherwise the triangle would be impossible (Art. 94); if it touches AB as at N, there will be only one triangle answering the conditions, viz. the right-angled triangle ACN; if it cuts AB in two points B, B', which must both be on the same side of A (since $CB < CA$) there are two triangles, viz. ACB and ACB' which have A, a, b, common; and the angles ABC, $AB'C$ are supplementary to one another, since $CB'B = CBB'$.

But if the given angle be obtuse, or if, the given angle being acute, the side opposite to it be greater than that adjacent to it, the points in which the circle cuts AN will be on opposite sides of A (fig. 20), and only the triangle CAB will have the proposed data; for the triangle CAB' will have the angle $CAB' = 180^0 - A$, instead of the given angle A.

106. This also appears from the formulæ of solution; for if M be the value of B less than 90^0 which satisfies the equation $\sin B = \frac{b}{a} \sin A$, the equation is equally satisfied by the obtuse angle $180^0 - M$ for the value of B.

Now if A be obtuse, B must be acute, and M is the value to be taken, and there is only one solution; and in order that the triangle may be possible, a must be greater than b.

Again, if A be acute, and $a > b$, A is greater than B; therefore B is acute, and M is the value to be taken, and the triangle is always possible.

But if A be acute, and $a < b$, then $A < B$, and this condition is satisfied both by the value M, and $180^0 - M$, for B; for, M being the value of B less than 90^0 which satisfies $\sin B = \frac{b}{a} \sin A$, A is less than M and *a fortiori* less than $180^0 - M$, since M is an acute angle; consequently there will be two solutions, provided $\sin B = \frac{b \sin A}{a}$ be a possible equation, or $a > b \sin A$.

The above methods may be verified upon the sides and angles of the following triangle.

$a = 9459{\cdot}31$	$\log a = 3{\cdot}9758595$,
$b = 8032{\cdot}29$	$\log b = 3{\cdot}9048394$,
$c = 8242{\cdot}58$	$\log c = 3{\cdot}9160632$,
$A =$ [illegible]$^\circ$. [illegible]$'$. $34''$	$L \sin A = 9{\cdot}9758250$,
$B =$ [illegible]$^\circ . 26' . 0'',38$	$L \sin B = 9{\cdot}9048049$,
$C = 55^\circ\ 30' . 24''.16$	$L \sin C = 9{\cdot}9160287$.

107. (3) Having given two sides a, b, and the included angle C, to find the other two angles, and the third side.

We have

$$\frac{a}{b} = \frac{\sin A}{\sin B};$$

$$\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B} \;\ldots\ldots (1)$$

$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B};$$

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

But (Art. 53) $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)};$$

$$\therefore \frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)};$$

$$\text{but } \tfrac{1}{2}(A+B) = 90^\circ - \frac{C}{2},$$

$$\text{and } \tan \tfrac{1}{2}(A+B) = \cot \frac{C}{2},$$

$$\therefore \tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{C}{2},$$

which gives $\frac{1}{2}(A-B)$; and since $\frac{1}{2}(A+B)$ is known, we have $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$ and $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)$; and then c is known from the equation

$$c = b\frac{\sin C}{\sin B} \;\ldots\ldots (2).$$

108. Dividing the last equation by equation (1) we find

$$\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B}.$$

$$\text{But } \sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2},$$

$$\text{and } \sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$$

$$= 2 \cos \frac{C}{2} \cos \tfrac{1}{2}(A - B), \quad \because \tfrac{1}{2}(A + B) = 90^0 - \tfrac{1}{2} C;$$

$$\therefore c = \frac{(a + b) \sin \frac{C}{2}}{\cos \frac{1}{2}(A - B)} \ldots\ldots(3),$$

a formula which will give c by the aid of only two additional logarithms (instead of three which equation (2) requires) since $\log (a + b)$ is already employed in the process.

109. When two sides and the included angle are given, the third side c has been found only by means of A and B previously determined. It may however be found directly from the formula

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

which must be adapted to logarithmic computation by means of a subsidiary angle. Among the different transformations which this formula may undergo, the following is the most remarkable. Since

$$\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} = 1, \quad \text{and } \cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} = \cos C,$$

$$c^2 = (a^2 + b^2)\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= (a + b)^2 \sin^2 \frac{C}{2} + (a - b)^2 \cos^2 \frac{C}{2}$$

$$= (a + b)^2 \sin^2 \frac{C}{2}\left\{1 + \left(\frac{a - b}{a + b} \cot \frac{C}{2}\right)^2\right\}.$$

Now since the tangent of an angle may have any value

$$\text{let } \tan\phi = \frac{a-b}{a+b}\cot\frac{C}{2},$$

$$\text{then } c = (a+b)\sin\frac{C}{2}\sqrt{1+\tan^2\phi}$$

$$\frac{(a+b)\sin\frac{C}{2}}{\cos\phi}$$

This solution only differs from the former one in appearance; for since ϕ is equal to $\frac{1}{2}(A-B)$, this value of c is identical with that furnished by equation (3).

110. If it should happen (as is often the case in practice) that a and b are given only by their logarithms, and that the angles A and B alone are required, then, instead of first finding a and b from the tables which the employment of the formula

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

would require, we may proceed thus.

Let ϕ be a subsidiary angle determined by the equation

$$\tan\phi = \frac{b}{a}, \qquad \text{or } L\tan\phi - 10 = \log b - \log a,$$

$$\text{then } \frac{a-b}{a+b} = \frac{1-\dfrac{b}{a}}{1+\dfrac{b}{a}} = \frac{1-\tan\phi}{1+\tan\phi} = \tan(45^0-\phi),$$

$$\therefore \tan\tfrac{1}{2}(A-B) = \tan(45^0-\phi)\cot\frac{C}{2}.$$

By this process for finding $\frac{1}{2}(A-B)$, there are two logarithms fewer to be looked out than if we first determine a and b.

111. (4) Having given the three sides, to find the angles. We have (Art. 93)

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

which gives A, and by similar formulæ may B and C be determined. But we must endeavour to find other formulæ more convenient for logarithms.

First, we have

$$2\sin^2\frac{A}{2} = 1 - \cos A,$$

and substituting for $\cos A$ its value, we get successively

$$2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2-b^2-c^2+2bc}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc},$$

$$\sin\frac{A}{2} = \sqrt{\frac{(a+b-c)(a-b+c)}{4bc}}.$$

To simplify this, let half the perimeter of the triangle be denoted by s, so that $a+b+c=2s$,

$$\therefore\ a+b-c = 2(s-c), \quad a-b+c = 2(s-b),$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \ \ldots\ldots\ldots\ldots (1).$$

Although the angle $\frac{A}{2}$ is determined by its sine, there can be no ambiguity; for A, being the angle of a triangle, is less than 180^0, and therefore $\frac{A}{2}$ less than 90^0.

112. With equal facility may similar expressions for $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$ be found. For

$$2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{b^2+c^2+2bc-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2bc} = \frac{2s \cdot 2(s-a)}{2bc},$$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \dots\dots\dots\dots\dots\dots\dots (2).$$

Also, dividing (1) by (2), we get

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \dots\dots\dots\dots\dots\dots (3).$$

113. As each of these formulæ requires four logarithms for its computation, when we want to find only one angle of a triangle, there is no reason for preferring one to another, except that when $\frac{1}{2}A$ is nearly 90^0, we must avoid determining it by its sine or tangent, and when it is very small, we must avoid determining it by its cosine; and that when $\frac{1}{2}A$ is less than 45^0, the tables will determine it from its sine with greater precision than from its cosine; and *vice versâ* when $\frac{1}{2}A$ is greater than 45^0 (Art. 85—87). When however we want to find two angles, the formula for the tangent of half an angle is to be preferred, as we shall need only the four logarithms of s, $s-a$, $s-b$, $s-c$; whereas, using either of the others, we shall need six logarithms.

114. A triangle, as we know, cannot be formed with three given sides, unless the sum of any two be greater than the third. This also appears from the preceding formulæ. Thus, taking $\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$, suppose $b > a + c$; $\therefore 2b > 2s$ or $s-b$ is negative, and $2s > 2(a+c)$, or $s - c > a$, and is therefore positive; the value of $\sin\frac{A}{2}$ is therefore imaginary; and similarly it may be shewn to be imaginary supposing $c > a + b$. If $a > b + c$, then $s > b + c$;

$$\therefore s-b > c,\ s-c > b,\ \therefore (s-b)(s-c) > bc,$$

or the value of $\sin\frac{A}{2}$ greater than 1, which is absurd. If b or c be supposed equal to the sum of the two others or to s, the value of $\sin\frac{A}{2}$ is evidently zero; if $a = b + c = s$, the value of $\sin\frac{A}{2}$ is 1; this is what ought to happen, as the triangle is reduced to a straight line.

115. Taking twice the product of the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$, we find

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

As this formula requires seven logarithms, it is not convenient for the calculation of A. It shews that $\frac{\sin A}{a}$ is a symmetrical expression

in terms of a, b, c, that is, remains the same when a, b, c are interchanged in any manner; therefore, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

conformable to the theorem of Art. 91, which is in this way perceived to be a consequence of the theorem of Art. 92. The latter theorem gives the above value of $\sin A$ immediately, but unresolved into factors, viz.

$$2bc \sin A = \sqrt{4b^2c^2 - (b^2+c^2-a^2)^2} = \sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}.$$

Method of determining heights and distances.

We shall now give some examples of the employment of Trigonometry in determining heights and distances. The problems which occur in practice may generally be reduced to one of the following.

116. (1) To find the height of an object the foot of which is accessible.

Measure along the ground, supposed horizontal, a base DE (fig. 21) from the foot of the object; and in order to avoid small angles, take it of a size neither very large nor very small compared with the height PD. Place at E the foot of the instrument for measuring angles, and measure the angle PAC formed by AP with the horizontal line AC parallel to DE. Then in the right-angled triangle PAC we know AC and the angle A; therefore we can calculate PC by Art. 98; and adding this to AE, we shall have the required height PD.

117. (2) To find the distance of the point in which an observer is placed from any visible but inaccessible point.

Measure a base AB (fig. 22) from A the place of the observer, and at each end observe the angle which the distance between the other end and the object P subtends, *i. e.* observe the angles BAP, ABP. Then in the triangle APB, one side and two angles being known, AP may be calculated by Art. 103.

118. (3) To find the height of an object whose foot is inaccessible.

By measuring a base AB (fig. 22) in any convenient direction from A the place of the observer, and proceeding as in the last problem, the distance AP may be determined; then in the triangle PAC, where AC is horizontal, AP being known, and the angle PAC observed, PC may be determined by Art. 97.

119. (4) To find the distance PQ between two visible but inaccessible points.

Measure a base AB (fig. 23), and at the station A observe the angles BAP, BAQ, and at the station B observe the angles ABQ, ABP. Then in each of the triangles APB, AQB, we know two angles and a side, and therefore can compute AP, AQ, a side of each; and if we observe the angle PAQ, or take the difference of PAB, QAB, if the four points P, Q, B, A, are in one plane, we shall know in the triangle APQ two sides and the included angle, and can therefore compute the third side PQ, by Art. 109.

120. (5) Three objects A, B, C being situated in a horizontal plane at known distances from one another, to determine a point M at which these distances are seen under given angles (fig. 24).

Since the angles AMB, AMC are known, if we describe on AB a segment containing an angle equal to the former, and on AC a segment containing an angle equal to the latter, the circles will intersect in A and in M, and M will be the point required. But this construction being impracticable in a survey, we shall shew how to calculate the angle BAM and the side AM; and first to find the angles ABM, ACM; assume the given quantities $AB=b$, $AC=c$, angle $BAC=A$, $AMB=B$, $AMC=C$, and the unknown quantities $ABM=x$, $ACM=y$. Then in the quadrilateral $ABMC$

$$x+y=360^\circ-(A+B+C),$$

therefore the sum of the angles x and y is known. Next, to find their difference. The triangles ABM, ACM give

$$AM=\frac{b\sin x}{\sin B},\quad AM=\frac{c\sin y}{\sin C}\ldots(1),$$

$$\therefore\ \frac{b\sin x}{\sin B}=\frac{c\sin y}{\sin C},\ \text{and}\ \frac{\sin x}{\sin y}=\frac{c\sin B}{b\sin C}.$$

Let $\tan\phi=\dfrac{b\sin C}{c\sin B}$, then ϕ can be calculated by logarithms;

$$\therefore\ \frac{\sin x}{\sin y}=\frac{1}{\tan\phi},\quad \frac{\sin x+\sin y}{\sin x-\sin y}=\frac{1+\tan\phi}{1-\tan\phi},$$

$$\text{or}\ \frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}=\tan(45^\circ+\phi)\ \text{(Arts. 47 and 53)};$$

therefore the difference $x-y$ is known, and consequently x and y.

Then angle $BAM=180^\circ-B-x$, and AM is known from either of the equations (1).

Instruments used in surveying.

121. Hadley's Sextant.

For measuring angular distances generally, this is much the most useful instrument that has ever been invented. The principle of its

construction, and the manner of using it, will be understood from what follows. Let FDE (fig. 29) be a graduated circular arc connected with its center C, by two bars CE, CF; and let B, C, be two plane reflectors, the former attached to the bar CE, and the latter to another bar CD, which is moveable about C and carries an index D that sweeps the graduations of the limb FE; on this account C is called the Index-glass. The reflectors are both perpendicular to the plane FCE, and are so placed that their surfaces are parallel when the index D is at O, and $a'b'$ is the position of ab; therefore the angle aCa' or OCD will measure their inclination when the moveable radius CD is in any other position. The upper part of the glass B is left transparent in order that objects may be seen directly through it, and by rays which pass close to the reflecting part. T is a telescope fixed to CF, having its optical axis parallel to the plane FCE, and directed to the line which separates the silvered from the unsilvered part of B.

When the angular distance of two objects S, P, is to be measured, the Sextant is held in such a position that its plane passes through them both; then the Telescope being directed to one of them P, so that it is seen by the direct ray PT, the radius CD, and consequently the mirror C which it carries, is moved till the other S is seen by a ray SC which, being incident on the mirror C, is reflected off at an equal angle into the direction CB, and again being incident on the silvered part of B, is reflected at an equal angle into the direction BT in which consequently the object S is seen, and therefore appears to coincide with P. Then if SC be produced to meet PT in H, since

$$\angle\, SCB = 180^\circ - 2\,BCa$$

$$\angle\, HBC = 180^\circ - 2\,CBc = 180^\circ - 2BCa';$$

$$\therefore \angle\, SHP = SCB - HBC = 2(aCa') = 2\,OCD;$$

therefore the angular distance of the objects is measured by twice the angle OCD. If therefore FE be graduated from O as the zero point, and each half degree be marked as a whole one, the reading, when the index is at D, will be the angular distance of S and P.

When the altitude of an object is to be taken, the plane of the instrument is held vertically, the telescope is directed to the point of the horizon immediately below the object so that TB is horizontal, and then the index is slided forward till the image of the object reflected from the index-glass, appears in contact with the horizon seen through the upper part of B, which on this account is called the Horizon-glass.

The Vernier.

122. As the index may not coincide exactly with one of the divisions of the limb of the Sextant, it is necessary to be able to estimate its distance from the preceding division, and this may be done by a contrivance called, from its inventor, a Vernier.

Suppose AB (fig. 30) to be a portion of the limb of the Sextant divided into equal parts each $=20'$, and let it be required to determine smaller portions, as for instance PQ. Let another circular arc whose length corresponds to that of 19 divisions of the limb, slide upon AB and be divided into 20 equal parts, each consequently $=19'$. Let its beginning, which is marked 0, coincide with Q, and observe which of its divisions coincides with a division of the limb. Suppose it that which is marked 2, then since the space between two divisions of the Vernier is less than the space between two divisions of the limb by $1'$, it follows that $PQ=2'$; and thus portions of $1'$, $2'$, &c. can be determined, although the limb itself is only divided into portions of $20'$.

Generally, a being the length of the space between two divisions of the limb, if $(n-1)a$ be the length of the Vernier and it be divided into n equal parts, and if, the zero of the Vernier being at Q, its m^{th} division coincide with a division of the limb, as at M,

$$\text{then } QM = m \text{ divisions of the Vernier} = m \cdot \frac{(n-1)a}{n},$$

$$\text{and } PM = m \text{ divisions of the limb} = ma,$$

$$\therefore PQ = \frac{ma}{n},$$

and is therefore known, since the number of minutes or seconds corresponding to $\frac{1}{n}a$ is known.

The Theodolite.

123. For the accurate measurement of horizontal and vertical angles, the Theodolite, which we next proceed to describe, may be considered as the most important instrument employed in surveying

It consists of two parallel circular plates A and B (fig. 31) which are together called the horizontal limb of the instrument. The lower one has the larger diameter, and has a slanting edge to receive the graduations; and underneath it is provided with foot-screws for making its plane horizontal when set upon the staff-head, and also a socket C at its middle into which descends the axis upon which the

upper circle is centered so as to turn with great steadiness and nicety upon the lower circle. The upper plate has its edge ground away at two opposite parts, a, a', so as to form at those parts a continued slanting surface with the edge of the lower plate; and on these are engraved the Verniers, for which reason the upper plate is called the Vernier-plate; and it carries two spirit levels at right angles to one another, their use being to determine when the horizontal limb is set level by altering the footscrews already mentioned for that purpose. Upon the Vernier-plate are erected two supports K, L, for the pivots of the horizontal axis of the circle MN to which the telescope ST is attached; to this axis the plane of the circle, as well as the optical axis of the telescope, are both at right angles; and the circle and telescope can turn completely round it without touching the Vernier-plate. The rim of the vertical circle is graduated, and subdivided by the help of two Verniers V, W, fixed to the ends of a bar held parallel to a diameter of the vertical circle by a stem rising from the Vernier-plate.

124. The following are the principal uses of the Theodolite.

(1) To measure the angle of elevation or depression of an object.

Turn the horizontal limb round its axis till the plane of the vertical circle passes through the object, and then elevate or depress the telescope till the object is seen on the cross wires, or in the direction of the optical axis of the telescope, and let TS (fig. 32) be that direction; also let Ovw be the rim which carries the divisions, and V, W, the zero points of its Verniers; read off the graduations of the rim which are opposite to V, W, that is, the number of degrees and minutes in the arcs Ov, Ovw, O being the beginning of the graduation of the rim. Next turn the horizontal limb through 180^0; then the plane of the vertical circle again passes through the object, but the optical axis of the telescope is in the direction $T'S'$ inclined to the horizon at the same angle as it was before; and therefore the angle $T'CT$ through which the vertical circle must now revolve to bring the object again upon the cross-wires of the telescope will be twice the zenith distance of the object; after this motion of the telescope let v', w', be the points of the rim opposite to V, W; then four times the zenith distance of the object

= twice the angle through which the circle has revolved

= sum of the angles subtended at the center of graduation by vv', ww', the arcs which have passed under the Verniers,

= difference of readings at V + diff. of readings at W.

(2) To measure the horizontal angle between two objects.

The instrument being placed exactly over the station from which the angle is to be taken, and the horizontality of the Vernier-plate being tested by noting that the bubbles in the spirit-levels remain stationary in the middle of their tubes, when the instrument is turned quite round; bring the object P (fig. 33) upon the cross-wires of the telescope, and read off the graduations of the horizontal limb which are opposite the zero points V and W of its Verniers, that is, the degrees and minutes in the arcs Ov, Ovw; next, by turning the instrument round its vertical axis, bring the other object Q upon the cross-wires, and read off the graduations of the arcs Ov', $Ov'w'$; then twice required angle $= 2\,PCQ$

= sum of angles subtended at the center of graduation by the arcs vv', ww', over which the Verniers have passed,

= difference of readings at V + diff. of readings at W.

Spirit Level.

125. The Spirit-level, in its simplest form, is a glass tube $ABCD$ (fig. 34) of uniform bore and of the form of a circular arc of very large radius. It is nearly filled with a fluid, such as ether, and the ends are closed, and if it be placed with its plane vertical and its extremities A and D in contact with a horizontal plane, the bubble BC of air left in the tube will be at the highest part of it; and if one end be gradually raised, the bubble will move towards that end.

In surveying, by a Level, is understood a telescope with a spirit-level attached to the upper or under side of its tube, and so adjusted that when the bubble is at the middle of the spirit-level, the optical axis of the telescope is horizontal; and it is mounted in a frame moveable round a vertical axis, so that the bubble preserves its position whilst the telescope is turned round horizontally on the staff-head.

To determine the difference of level of two places A and B (fig. 35), place a vertical staff at A, and let the instrument be set up at any convenient distance from A in a line towards B; and in the same line and at the same distance from the instrument as A is, set up another staff; then if the staves be divided into hundredths of a foot, with graduations and figures sufficiently large to be read by the observer, and if he first direct the telescope to A and read off the division of the staff bisected by the wires of the telescope, and then turn the telescope about, and read off the division similarly bisected on the second staff; the difference of these readings is the difference of elevation of the stations; and by continuing this process, the operation of levelling may be carried on for many miles.

Gunter's Chain.

126. In surveying, distances are usually measured by Gunter's Chain, which is 22 yards or four poles in length, and is divided into 100 links; consequently the length of each link is 7·92 inches. Hence since an acre contains 4840 square yards, it contains 10 square chains, or 100,000 square links; and therefore square links are converted into acres by cutting off five figures to the right.

Area of a triangle, quadrilateral, and regular polygon. Radii of their inscribed and circumscribed circles.

127. We shall now give the solutions of certain problems in Geometry, relating to the triangle, quadrilateral, and regular polygon, which, on account of the ease with which they are effected by Trigonometrical formulæ, usually form a part of treatises on this subject.

(1) Having given two sides and the included angle, or the three sides of a triangle, to find an expression for its area.

Let ABC be the triangle (fig. 18), and as one of the angles A, B is necessarily acute, let it be B; and let a perpendicular from C on the opposite side meet AB, or AB produced, in D; therefore in both cases $CD = b \sin A$. Then since the triangle is half the rectangle having the same base and altitude,

$$\text{area of the triangle} = \tfrac{1}{2} AB \times CD = \tfrac{1}{2} c \,.\, b \sin A\text{; or}$$

$$= \tfrac{1}{2} bc . \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} \ldots \text{(Art. 115)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

(2) Having given the three sides of a triangle, to find the radii of the circles circumscribed about it and inscribed in it.

Let CE (fig. 25) be the diameter of the circle circumscribed about the triangle ABC, CD a perpendicular on AB; join BE, then CBE is a right angle; and the angles CEB, CAB, being in the same segment, are equal to one another; or, if the perpendicular fall without the triangle, they are supplementary to one another;

$$\therefore \sin A = \sin CEB = \frac{CB}{CE}, \text{ or } CE = \frac{a}{\sin A},$$

that is, the diameter of the circumscribed circle is equal to the quotient of any side divided by the sine of the opposite angle.

Hence, substituting for $\sin A$ its value and dividing by 2, we find the radius of the circumscribed circle

$$\text{or } R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Next, if O be the center, and r the radius of the circle inscribed in the triangle ABC (fig. 26), and we join AO, BO, CO; then the triangle is divided into three others whose areas are equal to $\frac{1}{2}ar$, $\frac{1}{2}br$, $\frac{1}{2}cr$;

$$\therefore\ r.\tfrac{1}{2}(a+b+c) = \text{area of triangle } ABC,$$

$$\text{or } rs = \sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore\ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Also, it is easily seen that if O' be the center of the circle touching the side b, and the two other sides produced, and we join $O'A$, $O'B$, $O'C$, then from the two ways in which the quadrilateral $O'ABC$ may be made up, we have

$$\tfrac{1}{2}r'a + \tfrac{1}{2}r'c = \tfrac{1}{2}r'b + \text{area of triangle } ABC,$$

$$\therefore\ r'(s-b) = \sqrt{s(s-a)(s-b)(s-c)}.$$

128. (3) Having given the four sides of a quadrilateral whose opposite angles are supplementary to one another, to find its area and angles.

Let the sides $AB = a$, $BC = b$, $CD = c$, $AD = d$, (fig. 27), and the diagonals $AC = x$, $BD = y$; then from the triangles ABC, ADC, we have

$$2ab\cos B = a^2 + b^2 - x^2,$$

$$2dc\cos D = c^2 + d^2 - x^2,$$

$$\text{but } \cos D = \cos(180^0 - B) = -\cos B;$$

$$\therefore\ -2dc\cos B = c^2 + d^2 - x^2;$$

therefore, subtracting,

$$2(ab+cd)\cos B = a^2 + b^2 - c^2 - d^2;$$

$$\therefore\ \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2ab + 2cd}$$

$$1 + \cos B = \frac{a^2 + b^2 + 2ab - c^2 - d^2 + 2cd}{2ab + 2cd} = \frac{(a+b)^2 - (c-d)^2}{2(ab+cd)}$$

$$\text{or } 2\cos^2\frac{B}{2} = \frac{(a+b+c-d)(a+b-c+d)}{2(ab+cd)};$$

$$\text{and } 1-\cos B = \frac{c^2+d^2+2cd-a^2-b^2+2ab}{2(ab+cd,} = \frac{(c+d)^2-(a-b)^2}{2(ab+cd)}$$

$$\text{or } 2\sin^2\frac{B}{2} = \frac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}.$$

But area of quadrilateral $= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$

$$= \tfrac{1}{2}(ab+cd)\sin B = (ab+cd)\sin\frac{B}{2}\cos\frac{B}{2}$$

$$= \tfrac{1}{4}\sqrt{\{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)\}}$$

$$= \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}},$$

if half the perimeter $= \frac{1}{2}(a+b+c+d) = s$.

Also, any angle is known from the formula

$$\tan\frac{B}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}};$$

and since

$$x^2 = a^2+b^2-ab\frac{a^2+b^2-c^2-d^2}{ab+cd} = \frac{(ac+bd)(ad+bc)}{ab+cd},$$

the radius of the circumscribed circle, which is also circumscribed about the triangle ABC,

$$= \frac{x}{2\sin B} = \frac{x}{4\sin\frac{1}{2}B\cos\frac{1}{2}B} = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}.$$

129. (4) Having given the side of any regular polygon, to find its area, and the radii of the inscribed and circumscribed circles.

Let AB (fig. 28) be a side of a regular polygon, C the common center of the inscribed and circumscribed circles. Join AC, BC; draw CD perpendicular to AB, and consequently bisecting both AB and the angle ACB. Then if n be the number of sides, since each side subtends the same angle at C,

$$\angle ACB = \frac{360^0}{n}, \text{ and } \angle ACD = \frac{180^0}{n}.$$

Let $AB = a$, $AC = R$, $CD = r$; then in the right-angled triangle ACD

$$\frac{AD}{CD} = \tan\frac{180^0}{n}, \text{ or } r = \tfrac{1}{2}a\cot\frac{180^0}{n},$$

$$\frac{AD}{AC} = \sin\frac{180^0}{n}, \text{ or } R = \tfrac{1}{2}a\,\mathrm{cosec}\,\frac{180^0}{n};$$

and area of polygon $= n$ (area of triangle ACB)

$$= n\,.\,AD \times DC = n\frac{a^2}{4}\cot\frac{180^0}{n}.$$

Also, using the circular measure of two right angles, we have

$$\text{perimeter of polygon} = n\,.\,AB = 2n\,.\,r\tan\frac{\pi}{n} = 2\pi r\tan\frac{\pi}{n} \div \frac{\pi}{n};$$

$$\text{area of polygon} = n\,.\,CD \times AD$$

$$= nr\,.\,r\tan\frac{\pi}{n} = \pi r^2\tan\frac{\pi}{n} \div \frac{\pi}{n};$$

therefore, taking the limit of both sides when n is infinite, in which case the polygon becomes a circle whose radius is r, and the limit of $\tan\frac{\pi}{n} \div \frac{\pi}{n}$ is 1, (Art. 60), we get

circumference of a circle whose radius is $r = 2\pi r$,

area of a circle whose radius is $r = \pi r^2$.

SECTION V.

ON DEMOIVRE'S THEOREM, AND ON THE EXPONENTIAL EXPRESSIONS FOR THE SINE AND COSINE OF AN ANGLE.

Demoivre's Theorem.

130. THIS theorem, which goes by the name of its discoverer, is, that whatever be the index n, $\cos n\theta + \sqrt{-1} \sin n\theta$ is a value of $(\cos\theta + \sqrt{-1}\sin\theta)^n$.

It expresses that, in order to obtain a value of any power of the binomial $\cos\theta + \sqrt{-1}\sin\theta$, it is sufficient to multiply the angle θ by the index of the power. We may put, indifferently, the sign + or − before $\sqrt{-1}$, for that amounts to changing θ into $-\theta$.

We shall first consider the case where the index is a whole number. We get by multiplication

$$(\cos\theta+\sqrt{-1}\sin\theta)(\cos\phi+\sqrt{-1}\sin\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$+ \sqrt{-1}(\sin\theta\cos\phi + \cos\theta\sin\phi),$$

of which product, by the common formulæ, the part which is real $= \cos(\theta+\phi)$, and the imaginary part $= \sqrt{-1}\sin(\theta+\phi)$,

$$\therefore\ (\cos\theta + \sqrt{-1}\sin\theta)(\cos\phi + \sqrt{-1}\sin\phi) = \cos(\theta+\phi)$$
$$+ \sqrt{-1}\sin(\theta+\phi);$$

that is, the product of two factors of the form $\cos\theta + \sqrt{-1}\sin\theta$, is an expression of the same form, involving an angle equal to the sum of the angles of the factors. Hence, introducing another factor $\cos\psi + \sqrt{-1}\sin\psi$, we get

$$(\cos\theta+\sqrt{-1}\sin\theta)(\cos\phi+\sqrt{-1}\sin\phi)(\cos\psi+\sqrt{-1}\sin\psi)$$
$$= \{\cos(\theta+\phi) + \sqrt{-1}\sin(\theta+\phi)\}\ \{\cos\psi + \sqrt{-1}\sin\psi\}$$
$$= \cos(\theta+\phi+\psi) + \sqrt{-1}\sin(\theta+\phi+\psi),$$

and so on, to any number of factors; if, therefore, there be n factors all equal to $\cos\theta + \sqrt{-1}\sin\theta$, we shall have, since the first member admits but of one value,

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta \ldots (1).$$

Next, suppose the index to be negative; then, since

$$(\cos n\theta + \sqrt{-1}\sin n\theta)(\cos n\theta - \sqrt{-1}\sin n\theta)$$
$$= \cos^2 n\theta + \sin^2 n\theta = 1,$$

$$\therefore \frac{1}{\cos n\theta + \sqrt{-1}\sin n\theta} = \cos n\theta - \sqrt{-1}\sin n\theta,$$

or $$\frac{1}{(\cos\theta + \sqrt{-1}\sin\theta)^n} = \cos n\theta - \sqrt{-1}\sin n\theta,$$

or $(\cos\theta + \sqrt{-1}\sin\theta)^{-n} = \cos(-n\theta) + \sqrt{-1}\sin(-n\theta)$,
which proves the theorem for negative indices.

Lastly, suppose the index to be fractional. Replacing θ by $\frac{m\theta}{n}$ in equation (1), we get

$$\left(\cos\frac{m\theta}{n} + \sqrt{-1}\sin\frac{m\theta}{n}\right)^n = \cos m\theta + \sqrt{-1}\sin m\theta,$$
$$= (\cos\theta + \sqrt{-1}\sin\theta)^m, \text{ by the first case;}$$

therefore, extracting the n^{th} root of both sides, and employing a fractional index instead of a radical sign, we get

$$\cos\frac{m\theta}{n} + \sqrt{-1}\sin\frac{m\theta}{n}$$

for a value of

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

131. Hence it appears that $\cos\frac{m\theta}{n} + \sqrt{-1}\sin\frac{m\theta}{n}$ is one of the n different values which the expression

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}},$$

according to the principles of Algebra, admits of; and there is no difficulty in finding the remaining values.

For, by what has been proved, since θ is any angle whatever, and may, therefore, be replaced by the general value of all angles which have the same sine and cosine as θ, viz. $2r\pi + \theta$, r being any integer positive or negative, it follows that

$$\cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta)\ldots(2)$$

is a value of

$$\{\cos(2r\pi+\theta)+\sqrt{-1}\sin(2r\pi+\theta)\}^{\frac{m}{n}},$$

or of $$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}};$$

and we shall now shew that the expression (2) admits of n different values and no more.

First, if we make $r=0, 1, 2$, &c. $n-1$, we get n different values; for if two of them were alike, for instance when $r=p$ and $r=q$, it would be necessary that the angles $\frac{m}{n}(2p\pi+\theta)$, $\frac{m}{n}(2q\pi+\theta)$ should differ by a multiple of 2π, or that $\frac{m(p-q)\pi}{n}$ should be a multiple of π, which is impossible, since p and q are both less than n, and m not divisible by n. Also, if we take for r some number beyond the limits 0 and $n-1$, we shall get no new value; for suppose $r=\lambda n+r'$, where λ is any positive or negative number, and r' positive and $<n$, so that r may represent any positive or negative number whatever; then the above expression becomes

$$\cos\left\{2\lambda m\pi+\frac{m}{n}(2r'\pi+\theta)\right\}+\sqrt{-1}\sin\left\{2\lambda m\pi+\frac{m}{n}(2r'\pi+\theta)\right\},$$

or, suppressing the multiple of 2π,

$$\cos\frac{m}{n}(2r'\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r'\pi+\theta),$$

which, since r' is positive and less than n, is comprised among the values obtained in making $r=0, 1, 2, \ldots n-1$.

Consequently, the complete value of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$ is given by the equation

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}=\cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta),$$

r being any integer whatever, positive or negative, not excluding zero; and the n values of the first member result from the second, by taking r from 0 to $n-1$.

If we make $m=1$, the formula for extracting the n^{th} root of

$$\cos\theta+\sqrt{-1}\sin\theta \text{ is}$$

$$\sqrt[n]{(\cos\theta+\sqrt{-1}\sin\theta)}=\cos\frac{2r\pi+\theta}{n}+\quad 1\sin\frac{2r\pi+\theta}{}$$

and putting $\theta=0$, $\theta=\pi$, the general values of $(1)^m$ and $(-1)^m$ become, respectively,

$\cos 2rm\pi+\sqrt{-1}\sin 2rm\pi$, $\quad \cos(2r+1)m\pi+\sqrt{-1}\sin(2r+1)m\pi$,

where m is supposed to be fractional.

Obs. Since $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$ may be supposed to mean either

$$\{\sqrt[n]{(\cos\theta+\sqrt{-1}\sin\theta)}\}^m=\left\{\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}\right\}^m,$$

$$\text{or } \sqrt[n]{(\cos\theta+\sqrt{-1}\sin\theta)^m}=\sqrt[n]{(\cos m\theta+\sqrt{-1}\sin m\theta)},$$

the values corresponding to these two suppositions must be identical:

$$\text{viz. } \cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta),$$

$$\text{and } \cos\frac{1}{n}(2r\pi+m\theta)+\sqrt{-1}\sin\frac{1}{n}(2r\pi+m\theta).$$

Now, taking r from 1 to $n-1$, the multipliers of 2π involved in the former will be

$$\frac{m}{n},\ \frac{2m}{n},\ \frac{3m}{n},\ \&c.,\ \frac{(n-1)m}{n};$$

and if the division be performed in each fraction, no two remainders can be alike; for suppose $\frac{pm}{n}$ and $\frac{qm}{n}$ to give the same remainder, then $(p-q)\frac{m}{n}$ is a whole number, which is absurd since p and q are both less than n; therefore the remainders will produce the terms of the series 1, 2, 3, &c. $(n-1)$; and therefore, suppressing the multiples of 2π, the n values of the former expression will be identical with those of the latter.

Formulæ for expressing the sine and cosine of the sum of any angles, or of a multiple angle, in terms of the sines and cosines of the simple angles.

132. To express the sine and cosine of the sum of any number of angles, in terms of the sines and cosines of the angles; or the tangent of the sum, in terms of the tangents of the angles.

From Art. 130 it appears that

$$\cos(\theta+\phi+\&c.+\lambda)+\sqrt{-1}\sin(\theta+\phi+\&c.+\lambda)$$
$$=\cos\theta\cos\phi\ldots\cos\lambda$$
$$\times(1+\sqrt{-1}\tan\theta)(1+\sqrt{-1}\tan\phi)\ldots(1+\sqrt{-1}\tan\lambda).$$

If, therefore, we effect the multiplication of the factors of the second member, and by s_1, s_2, s_3, &c. denote the sum of the tangents of θ, ϕ, &c. λ, the sum of the products of every two of these tangents, the sum of the products of every three, &c.; and by σ the sum of the angles θ, ϕ, &c. λ, of which we suppose the number to be n, we shall have (Theory of Equations, Art. 19)

$$\cos\sigma + \sqrt{-1}\sin\sigma = \cos\theta\cos\phi\ldots\ldots\cos\lambda\left\{1 + \sqrt{-1}\,s_1 - s_2 - \sqrt{-1}\,s_3 + s_4 + \sqrt{-1}\,s_5 - \&\text{c.} + (\sqrt{-1})^n s_n\right\}.$$

Therefore, equating possible and impossible parts,

$$\sin\sigma = \cos\theta\cos\phi\ldots\ldots\cos\lambda\left\{s_1 - s_3 + s_5 - \&\text{c.}\right\},$$

$$\cos\sigma = \cos\theta\cos\phi\ldots\ldots\cos\lambda\left\{1 - s_2 + s_4 - \&\text{c.}\right\};$$

and, dividing the upper equation by the lower,

$$\tan\sigma = \frac{s_1 - s_3 + s_5 - \&\text{c.}}{1 - s_2 + s_4 - \&\text{c.}}.$$

If n be odd, the numerator will be continued to s_n, and the denominator to s_{n-1}; and *vice versâ*, if n be even.

Hence $\tan\sigma$ is expressed in terms of the tangents of θ, ϕ, &c. λ; and it is evident that the values of $\sin\sigma$ and $\cos\sigma$, if each term of the series within brackets be multiplied by the factor without the brackets, will contain only the sines and cosines of θ, ϕ, &c. λ. If we suppose ϕ, ψ, &c. λ, to be all equal to one another and to θ, we get the values of the sine, cosine, and tangent of $n\theta$; but it is better to obtain those values directly, as follows.

133. Resuming *Demoivre's* formula, and supposing n a positive integer, we have

$$\cos n\theta + \sqrt{-1}\sin n\theta = (\cos\theta + \sqrt{-1}\sin\theta)^n;$$

but the second member expanded by the binomial theorem gives

$$\cos^n\theta + \frac{n}{1}\cos^{n-1}\theta\sqrt{-1}\sin\theta - \frac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\sin^2\theta$$

$$-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cos^{n-3}\theta\sqrt{-1}\sin^3\theta + \&\text{c.};$$

therefore, equating possible and impossible parts,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \&c.$$

$$\sin n\theta = \frac{n}{1}\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos^{n-5}\theta\sin^5\theta - \&c.$$

These formulæ express the sine and cosine of the multiple angle $n\theta$ in terms of the sine and cosine of the simple angle; their law is evident, and, like the binomial theorem from which they are deduced, each is to be continued till we arrive at a term zero.

134. The above formulæ may be transformed so as to involve cosines only, or sines only, in the following manner.

Suppose n even, then the general term of the series for $\cos n\theta$ is

$$(-1)^r\frac{n(n-1)\ldots(n-2r+1)}{1.2.3\ldots 2r}\sin^{2r}\theta\,(1-\sin^2\theta)^{\frac{n}{2}-r},$$

where r begins from zero; therefore, making $r = 0, 1, 2, 3,$ &c. successively, we get

$$\cos n\theta = 1 - \frac{n}{2}\sin^2\theta + \frac{n(n-2)}{2.4}\sin^4\theta - \frac{n(n-2)(n-4)}{2.4.6}\sin^6\theta + \&c.$$

$$-\frac{n(n-1)}{1.2}\sin^2\theta\left\{1-\frac{n-2}{2}\sin^2\theta + \frac{(n-2)(n-4)}{2.4}\sin^4\theta - \&c.\right\}$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$$

$$\times\sin^4\theta\left\{1-\frac{n-4}{2}\sin^2\theta + \frac{(n-4)(n-6)}{2.4}\sin^4\theta - \&c.\right\};$$

$$\text{or }\cos n\theta = 1 - \frac{n}{1}\left(\frac{1}{2}+\frac{n-1}{2}\right)\sin^2\theta + \frac{n(n-2)}{1.3}\left\{\frac{3.1}{2.4}+\frac{3}{2}\frac{n-1}{2}\right.$$

$$\left.+\frac{(n-1)(n-3)}{2.4}\right\}\sin^4\theta - \&c.$$

$$\text{but }\frac{1}{2}+\frac{n-1}{2} = \frac{n}{2},$$

$$\frac{3.1}{2.4}+\frac{3}{2}\cdot\frac{n-1}{2}+\frac{(n-1)(n-3)}{2.4}=\frac{n(n+2)}{2.4},$$

and, in general, as appears by equating the coefficient of x^r in the product of the expansions of $(1+x)^{\frac{m}{2}}$ and $(1+x)^{\frac{n}{2}}$, and in the equivalent expansion of $(1+x)^{\frac{m+n}{2}}$,

$$\frac{m(m-2)\ldots(m-2r+2)}{2.4\ldots 2r}+\frac{m(m-2)\ldots(m-2r+4)}{2.4\ldots(2r-2)}\frac{n}{2}$$

$$+\frac{m(m-2)\ldots(m-2r+6)}{2.4\ldots(2r-4)}\cdot\frac{n(n-2)}{2.4}+\&c.$$

$$=\frac{(m+n)(m+n-2)\ldots(m+n-2r+2)}{2.4.6\ldots 2r};$$

$$\therefore\ \cos n\theta=1-\frac{n^2}{1.2}\sin^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta$$

$$-\frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6}\sin^6\theta+\&c.\quad(1)$$

Exactly in the same manner, n being even, we get

$$\sin n\theta=n\cos\theta\left\{\sin\theta-\frac{n^2-2^2}{1.2.3}\sin^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\sin^5\theta-\&c.\right\};$$

and, when n is odd, the formulæ are

$$\cos n\theta=\cos\theta\left\{1-\frac{n^2-1}{1.2}\sin^2\theta+\frac{(n^2-1)(n^2-3^2)}{1.2.3.4}\sin^4\theta-\&c.\right\},$$

$$\sin n\theta=n\sin\theta\left\{1-\frac{n^2-1}{1.2.3}\sin^2\theta+\frac{(n^2-1)(n^2-3^2)}{1.2.3.4.5}\sin^4\theta-\&c.\right\}.\quad(2)$$

If in these four formulæ we replace θ by $\frac{\pi}{2}-\theta$, we get others for $\cos n\theta$ and $\sin n\theta$ proceeding according to powers of $\cos\theta$; viz. when n is even,

$$(-1)^{\frac{n}{2}}\cos n\theta=1-\frac{n^2}{1.2}\cos^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta-\&c.,\quad(3)$$

$$(-1)^{\frac{n}{2}+1}\sin n\theta=n\sin\theta\left\{\cos\theta-\frac{n^2-2^2}{1.2.3}\cos^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\cos^5\theta-\&c.\right\};$$

and when n is an odd integer,

$$(-1)^{\frac{n-1}{2}}\cos n\theta=n\cos\theta-\frac{n(n^2-1^2)}{1.2.3}\cos^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\cos^5\theta-\&c.,\ (4)$$

$$(-1)^{\frac{n-1}{2}}\sin n\theta=\sin\theta\left\{1-\frac{n^2-1^2}{1.2}\cos^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\cos^4\theta-\&c.\right\}.$$

If in equations 1, 2, 3, 4 we denote by S_1, S_2, S_3, S_4 the series forming the second members which are expressed entirely in sines and cosines, then when n is fractional the values of $\cos n\theta$ and $\sin n\theta$ in ascending powers of $\sin\theta$ (r being any integer) are

$$\cos n\theta = \cos nr\pi \,.\, S_1 - \sin (n-1)\, r\pi \,.\, S_2,$$

$$\sin n\theta = \sin nr\pi \,.\, S_1 + \cos (n-1)\, r\pi \,.\, S_2;$$

and in ascending powers of $\cos\theta$,

$$\cos n\theta = \cos n\,(2r+1)\tfrac{1}{2}\pi \,.\, S_3 + \cos (n-1)(2r+1)\tfrac{1}{2}\pi \,.\, S_4,$$

$$\sin n\theta = \sin n\,(2r+1)\tfrac{1}{2}\pi \,.\, S_3 + \sin (n-1)(2r+1)\tfrac{1}{2}\pi \,.\, S_4.$$

The investigation of these series by the method of Indeterminate coefficients, may be best deferred till the Student arrives at the Differential Calculus.

For the resolution of $\cos n\theta$ and $\sin n\theta$ into their factors, recourse may be had to Theory of Equations, Art. 22.

135. A descending series for $\cos n\theta$ in terms of $\cos\theta$ may be investigated more simply in the following manner.

Since $\quad 1 - pz + z^2 = (1 - xz)\left(1 - \dfrac{z}{x}\right)$ if $p = x + \dfrac{1}{x}$,

$$\text{or } 1 - z(p - z) = (1 - xz)\left(1 - \frac{z}{x}\right),$$

taking the *Napierian* logarithm of both sides, and writing down only the terms which when developed will involve z^n, we have

$$\frac{z^n}{n}(p-z)^n + \frac{z^{n-1}}{n-1}(p-z)^{n-1} + \frac{z^{n-2}}{n-2}(p-z)^{n-2} + \&c.$$

$$= \frac{z^n}{n}\left(x^n + \frac{1}{x^n}\right) + \&c.;$$

therefore, equating the coefficients of z^n in the two members of this equation,

$$\frac{1}{n}p^n + \frac{1}{n-1}\{-(n-1)\,p^{n-2}\} + \frac{1}{n-2}\left\{\frac{(n-2)(n-3)}{1\,.\,2}\,p^{n-4}\right\} + \&c.$$

$$+ \frac{1}{n-r}\left\{(-1)^r \frac{(n-r)(n-r-1)\ldots(n-2r+1)}{1\,.\,2\,.\,3\ldots r}\,p^{n-2r}\right\} + \&c.$$

$$= \frac{1}{n}\left(x^n + \frac{1}{x^n}\right).$$

But if $2\cos\theta = x + \dfrac{1}{x} = p$, then $2\cos n\theta = x^n + \dfrac{1}{x^n}$, and $(2\cos\theta)^n = p^n$,

$$\therefore\ 2\cos n\theta = (2\cos\theta)^n - n\,(2\cos\theta)^{n-2} + \frac{n(n-3)}{1\,.\,2}(2\cos\theta)^{n-4} - \&c.$$

$$+ (-1)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{1 . 2 . 3 \ldots r} (2\cos\theta)^{n-2r} + \&c.$$

In calculating by this formula we must follow the law indicated by the first terms, and stop at and exclude the first negative power of $2\cos\theta$.

136. If we write the formulæ of Art. 133, so as to have $(\cos\theta)^n$ a factor of the second members, we get

$$\sin n\theta = (\cos\theta)^n \left\{\frac{n}{1}\tan\theta - \frac{n(n-1)(n-2)}{1 . 2 . 3}\tan^3\theta + \&c.\right\}$$

$$\cos n\theta = (\cos\theta)^n \times$$

$$\left\{1 - \frac{n(n-1)}{1 . 2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}\tan^4\theta - \&c.\right\}$$

therefore, dividing the former equation by the latter, we get $\tan n\theta$ expressed in terms of $\tan\theta$, viz.

$$\tan n\theta = \frac{\frac{n}{1}\tan\theta - \frac{n(n-1)(n-2)}{1 . 2 . 3}\tan^3\theta + \&c.}{1 - \frac{n(n-1)}{1 . 2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}\tan^4\theta - \&c.}$$

where each series has its terms alternately positive and negative, and is to be continued till we arrive at a term zero.

137. We shall now shew how *Euler*, from the above series for $\sin n\theta$ and $\cos n\theta$ (Art. 133), deduces the expansions of the sine and cosine of an angle in terms of the circular measure of the angle; for it is no longer indifferent whether the angle is expressed by degrees, &c., or by its circular measure, as is the case when only the trigonometrical ratios of the angle are involved.

We may, without departing from the hypothesis of n an integer, dispose of θ so that $n\theta$ shall be equal to any given angle a. Let therefore $n\theta = a$, or $n = a \div \theta$; then with this value the formulæ become

$$\cos a = (\cos\theta)^n - \frac{a(a-\theta)}{1 . 2}(\cos\theta)^{n-2}\left(\frac{\sin\theta}{\theta}\right)$$

$$+\frac{a(a-\theta)(a-2\theta)(a-3\theta)}{1.2.3.4}(\cos\theta)^{n-4}\left(\frac{\sin\theta}{\theta}\right)^4 - \&c.$$

$$\sin a = \frac{a}{1}(\cos\theta)^{n-1}\left(\frac{\sin\theta}{\theta}\right) \quad \frac{a(a-\theta)(a-2\theta)}{1.2.3}(\cos\theta)^{n-3}\left(\frac{\sin\theta}{\theta}\right)^3$$
$$+ \&c.$$

Conceive now θ to diminish to zero, and n to increase to infinity; then these formulæ will exhibit no further traces of θ and of n, and will contain a only. For when $\theta = 0$, $\cos\theta = 1$, and $\sin\theta \div \theta = 1$, (Art. 60); and for that value of θ, we may also admit that the powers of $\cos\theta$ and of $\sin\theta \div \theta$ are equal to unity, however great be the indices of the powers; consequently the above formulæ become

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \frac{a^6}{1.2.3.4.5.6} + \&c.$$

$$\sin a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&c.$$

As the number n has become infinite, these series do not terminate; but they are not the less proper to give very approximate values of the sine and cosine when a is a small fraction; and they are always convergent for any value of a.

138. By dividing three terms of the lower series by three terms of the upper, we obtain the first three terms of the expansion of $\tan a$, viz.

$$\tan a = a + \frac{a^3}{3} + \frac{2a^5}{3.5} + \frac{17a^7}{3.3.5.7} + \&c;$$

but the law of the series is not easily discoverable in this way. Similarly, by dividing unity by each of these series, the first 3 or 4 terms of the expansions of $\sec a$, $\operatorname{cosec} a$, $\cot a$, may be obtained.

Formulæ for expressing the powers of the sine or cosine of an angle in terms of the sines or cosines of its multiples.

139. In the higher branches of Mathematics it is frequently necessary to express the powers of the sine or cosine of an angle in terms of the sines or cosines of multiples of the angle. When the index is a positive integer, which is the ordinary case, it may be effected in the following manner.

If we assume

$$\cos\theta + \sqrt{-1}\sin\theta = x,$$

then since

$$(\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta - \sqrt{-1}\sin\theta) = \cos^2\theta + \sin^2\theta = 1,$$

$$\cos\theta - \sqrt{-1}\sin\theta = \frac{1}{x};$$

therefore, by adding and subtracting, we get

$$2\cos\theta = x + \frac{1}{x}, \qquad 2\sqrt{-1}\sin\theta = x - \frac{1}{x}.$$

Also, by *Demoivre's* theorem,

$$\cos n\theta + \sqrt{-1}\sin n\theta = x^n,$$

$$\cos n\theta - \sqrt{-1}\sin n\theta = \frac{1}{x^n},$$

$$\therefore\ 2\cos n\theta = x^n + \frac{1}{x^n}, \qquad 2\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}.$$

Hence

$$(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n = x^n + nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} + \&c.$$

$$+ \frac{n(n-1)}{1.2}\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} + \frac{1}{x^n},$$

or, grouping together the terms equidistant from the beginning and end which have the same coefficients,

$$(2\cos\theta)^n = x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1.2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

If n be even, the number of terms in the expansion of $\left(x + \frac{1}{x}\right)^n$ is $n + 1$, which is odd; there will therefore be one term, viz. the $(\frac{1}{2}n + 1)^{\text{th}}$, or middle term, which has no fellow; it will be

$$\frac{n(n-1)(n-2)\ldots(n-\frac{1}{2}n+1)}{1.2.3\ldots\frac{1}{2}n}x^{\frac{n}{2}}.\frac{1}{x^{\frac{n}{2}}}, \text{ or } \frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2.3\ldots\frac{1}{2}n}.$$

If n be odd, the number of terms will be $n+1$ which is even, there will therefore be an exact number, $\frac{1}{2}(n+1)$, of pairs of terms; and the last pair consisting of the two middle terms of the expanded binomial, *i.e.* of the $\{\frac{1}{2}(n-1)+1\}^{\text{th}}$ and $\{\frac{1}{2}(n+1)+1\}^{\text{th}}$ terms, will be

$$\frac{n(n-1)\ldots\{n-\frac{1}{2}(n-1)+1\}}{1.2.3\ldots\frac{1}{2}(n-1)}\left(x+\frac{1}{x}\right)$$

$$\text{or } \frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2.3\ldots\frac{1}{2}(n-1)}\left(x+\frac{1}{x}\right).$$

Hence, replacing $x^n+\frac{1}{x^n}$ by $2\cos n\theta$, &c. and dividing by 2, we get

$$2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{1.2}\cos(n-4)\theta+\&c.$$

the last term being, according as n is even or odd,

$$\frac{1}{2}\cdot\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2.3\ldots\frac{1}{2}n},\text{ or }\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2.3\ldots\frac{1}{2}(n-1)}\cos\theta;$$

that is, we must stop at and exclude the term which involves the first negative angle; and we must take only half of the last term when it involves the angle zero.

140. Again, to express $(\sin\theta)^n$ in terms of sines or cosines of multiples of θ. We have

$$\therefore (2\sqrt{-1})^n\sin^n\theta=\left\{x+\frac{-1}{x}\right\}^n=x^n-nx^{n-2}+\frac{n(n-1)}{1.2}x^{n-4}-\&c.$$

$$\frac{n(n-1)}{1.2}x^2\left(\frac{-1}{x}\right)^{n-2}+nx\left(\frac{-1}{x}\right)^{n-1}+\left(\frac{-1}{x}\right)^n.$$

First, let n be even; then the first member will equal $2^n(-1)^{\frac{n}{2}}\sin^n\theta$, and the second member will equal

$$x^n+\frac{1}{x^n}-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+\&c;$$

and, as in the case of the cosine, there will be a middle term

$$\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2.3\ldots\frac{1}{2}n}x^{\frac{n}{2}}\left(\frac{-1}{x}\right)^{\frac{n}{2}},\text{ not involving } x;$$

therefore, replacing $x^n + \frac{1}{x^n}$ by $2 \cos n\theta$, &c. and dividing by 2,

$$2^{n-1} (-1)^{\frac{n}{2}} \sin^n\theta = \cos n\theta - n \cos (n-2)\theta$$
$$+ \frac{n(n-1)}{1.2} \cos (n-4)\theta - \&c.$$
$$+ \tfrac{1}{2} (-1)^{\frac{n}{2}} \frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2.3\ldots\frac{1}{2}n}.$$

If n be odd, the first member will be $2^n \sqrt{-1} (-1)^{\frac{n-1}{2}} \sin^n\theta$, and the second member will be $x^n - \frac{1}{x^n} - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \&c.$; and, as in the case of the cosine, the two middle terms will be

$$(-1)^{\frac{n-1}{2}} \frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2.3\ldots\frac{1}{2}(n-1)} \left(x - \frac{1}{x}\right);$$

therefore, replacing $x^n - \frac{1}{x^n}$ by $2\sqrt{-1} \sin n\theta$, &c. and dividing by $2\sqrt{-1}$, we find

$$2^{n-1} (-1)^{\frac{n-1}{2}} \sin^n\theta = \sin n\theta - n \sin (n-2)\theta$$
$$+ \frac{n(n-1)}{1.2} \sin (n-4)\theta - \&c.$$
$$+ (-1)^{\frac{n-1}{2}} \frac{n(n-1)\ldots\frac{1}{2}(n+3)}{1.2.3\ldots\frac{1}{2}(n-1)} \sin\theta.$$

In employing these formulæ for the purpose of calculating $\sin^n\theta$, we have only to follow the law indicated by the first terms, and to stop at and exclude the term which involves the first negative angle; and we must take only half of the coefficient of the last term when it involves the angle zero.

The results in this and the preceding Art. are evidently imperfect when n is fractional, as the second members have only one value, whilst the first members admit of several. The correct forms of the series for the case of n fractional, will be given hereafter.

Inverse Trigonometrical functions of an angle.

141. Since the sine, cosine, &c. of an angle do not increase indefinitely with the angle, but vary within certain limits

only, so that whatever values they have when the angle equals a, they receive the same when the angle equals $a + 2\pi$, $a + 4\pi$, &c., they are called *periodic* quantities to distinguish them from continually increasing quantities.

We have hitherto chiefly confined our reasonings to the case where only the sines, cosines, &c. of known angles enter into the calculation; but it is frequently necessary to consider the case where the angles themselves, as determined by their sines, cosines, &c. are introduced; and there is, as has been stated, an essential difference between these two cases; for in the former the quantities have each only a single value, whereas in the latter they have an infinite number of values. Thus $\sin 30^0$ has only one value, viz. $\frac{1}{2}$; but the angle whose sine $= \frac{1}{2}$, has an infinite number of values comprised in the general expression (Art. 31)

$$n\pi + (-1)^n \frac{\pi}{6}.$$

The notation usually employed to express an angle whose sine is x, an angle whose cosine is x, &c. is $\sin^{-1}x$, $\cos^{-1}x$, &c.; they are called *inverse* Trigonometrical Functions of the angle, in the same way as the Trigonometrical Ratios themselves are called direct Trigonometrical functions of the angle.

142. The reason for the above notation is the following. If an operation denoted by f be performed upon a quantity x, so that the result of it is $f(x)$, and if the same operation be now performed upon $f(x)$ as was performed upon x, the result will be $f\{f(x)\}$ or $ff(x)$, which may be written $f^2(x)$. Similarly $f[f\{f(x)\}]$ or $fff(x)$ may be written $f^3(x)$, and so on, which gives in general $f^m f^n(x) = f^{m+n}(x)$. To preserve the same equation, $f^0(x)$ must mean x simply; for making $n=0$, $m=1$, $f\{f^0(x)\} = f(x)$, and consequently $f^0(x) = x$; and if we now make $m=1$, $n=-1$, we shall discover the meaning of $f^{-1}(x)$; for

$$f\{f^{-1}(x)\} = f^0(x) = x,$$

and therefore $f^{-1}(x)$ means that function of x upon which if the operation denoted by f be performed, the result is x, or that function whose effect is exactly reversed by f; that is, $f^{-1}(x)$ denotes the inverse function of $f(x)$.

We have seen in the foregoing pages that although for any assigned value of the angle there is but one value of the sine, cosine,

&c., yet for any assigned value of the sine, cosine, &c. there is an infinite number of corresponding angles; the reason, we may repeat, being that in the first four quadrants there are two distinct angles which have the same sine, cosine, or tangent; and that any multiple of 2π added or subtracted, does not alter the sine, cosine, or tangent. Hence, Art. 31-33, if α be the least positive angle which satisfies the equations

$$\sin\theta = x, \ \cos\theta = x, \ \tan\theta = x,$$

we have, respectively, for the values of the inverse functions,

$$\theta \text{ or } \sin^{-1} x = n\pi + (-1)^n \alpha,$$

$$\theta \text{ or } \cos^{-1} x = 2n\pi \pm \alpha, \quad \theta \text{ or } \tan^{-1} x = n\pi + \alpha,$$

n being any positive or negative integer whatever, not excluding zero. It is usual to take for the values of the inverse Trigonometrical functions, the least corresponding positive angles; but the multiplicity of their values must never be lost sight of.

143. From the above mode of expressing the inverse Trigonometrical functions arise the following formulæ, which are sometimes useful.

$$\text{Let} \quad \tan\theta = x, \qquad \tan\phi = y,$$

$$\text{then } \theta = \tan^{-1} x, \quad \phi = \tan^{-1} y;$$

$$\text{now } \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x+y}{1-xy},$$

$$\therefore \ \theta + \phi = \tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

$$\text{or } \tan^{-1} x + \tan^{-1} y = \tan^{-} \frac{x+y}{1-xy}.$$

$$\text{Similarly, } \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}.$$

And in the same way may these formulæ be deduced,

$$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right),$$

$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

Exponential expressions for the sine and cosine of an angle, and their consequences.

144. To investigate the exponential expressions for $\sin\theta$, $\cos\theta$, $\tan\theta$.

Since (Art. 26, Appendix), e being the base of *Napier's* system of logarithms,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

if we change x successively into $\theta\sqrt{-1}$, and $-\theta\sqrt{-1}$, we get

$$e^{\theta\sqrt{-1}} = 1 + \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} - \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} + \&c.$$

$$e^{-\theta\sqrt{-1}} = 1 - \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} + \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} - \&c.$$

therefore, adding and subtracting,

$$e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}} = 2\left(1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.\right)$$

$$e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}} = 2\sqrt{-1}\left(\theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c.\right)$$

and comparing these series with the values of $\cos\theta$ and $\sin\theta$, (Art. 137), we have

$$2\cos\theta = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$$

$$2\sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}},$$

and consequently by division

$$\tan\theta = \frac{1}{\sqrt{-1}}\,\frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}} = \frac{1}{\sqrt{-1}}\,\frac{e^{2\theta\sqrt{-1}} - 1}{e^{2\theta\sqrt{-1}} + 1}.$$

Also, adding and subtracting and dividing by 2, we get the important formulæ

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}$$

$$\cos\theta - \sqrt{-1}\sin\theta = e^{-\theta\sqrt{-1}},$$

the latter evidently resulting from the former by changing the sign of θ.

Moreover from Art. 131, m being fractional, we get for $(1)^m$ and $(-1)^m$, the recipients of the multiple values of any positive or negative quantity that is raised to the m^{th} power, the exponential expressions

$$(1)^m = e^{2rm\pi\sqrt{-1}}, \qquad (-1)^m = e^{(2r+1)m\pi\sqrt{-1}}.$$

145. To develope the circular measure of an angle in terms of the tangent of the angle.

We get from the preceding Art.

$$\epsilon^{2\theta\sqrt{-1}} = \frac{\cos\theta + \sqrt{-1}\sin\theta}{\cos\theta - \sqrt{-1}\sin\theta} = \frac{1+\sqrt{-1}\tan\theta}{1-\sqrt{-1}\tan\theta}:$$

$$\therefore\ 2\theta\sqrt{-1} = \log\frac{1+\sqrt{-1}\tan\theta}{1-\sqrt{-1}\tan\theta}$$

$$= 2\left\{\sqrt{-1}\tan\theta + \frac{1}{3}(\sqrt{-1}\tan\theta)^3 + \frac{1}{5}(\sqrt{-1}\tan\theta)^5 + \&c.\right\},$$

(Art. 31, Appendix)

$$\therefore\ \theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \&c.$$

146. Hence, by dividing half a right angle, whose tangent is unity, into two or more angles whose tangents are rational proper fractions, we can compute the circular measure of half a right angle, which is $\frac{\pi}{4}$.

For if α and β be two angles whose tangents are respectively $\frac{1}{2}$ and $\frac{1}{3}$, we have

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = 1 = \tan\frac{\pi}{4};$$

$$\therefore\ \frac{\pi}{4} = \alpha+\beta = \frac{1}{2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{5}\frac{1}{2^5} - \&c. + \frac{1}{3} - \frac{1}{3}\frac{1}{3^3} + \frac{1}{5}\frac{1}{3^5} - \&c.$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) - \&c.$$

147. But a series for the value of π, which converges far more rapidly, may be obtained as follows.

Since $\tan\frac{\pi}{4} = 1$, we find (Art. 48) $\tan\frac{\pi}{8} = \sqrt{2} - 1$, and

$\tan\frac{\pi}{16} = \cdot1989 = \frac{1}{5}$ nearly; if therefore $\tan\alpha = \frac{1}{5}$, 4α is a little greater than $\frac{\pi}{4}$, and we shall have $4\alpha = \frac{\pi}{4} + \beta$, where β is a small angle.

$$\text{Now } \tan 2\alpha = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12},\quad \tan 4\alpha = \frac{\frac{5}{6}}{1-\frac{25}{144}} = \frac{120}{119},$$

$$\therefore \tan\beta = \tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\frac{120}{119}-1}{\frac{120}{119}+1} = \frac{1}{239};$$

$$\therefore \frac{\pi}{4} = 4\alpha - \beta = 4\left(\frac{1}{5} - \frac{1}{3}\frac{1}{5^3} + \frac{1}{5}\frac{1}{5^5} - \&c.\right)$$
$$- \left\{\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3 + \&c.\right\},$$

from which $\pi = 3\cdot1415926535$ may be rapidly obtained, since the former series may be written

$$\frac{8}{10} - \frac{1}{3}\cdot\frac{32}{10^3} + \frac{1}{5}\cdot\frac{128}{10^5} - \&c. = \cdot8 - \frac{1}{3}(\cdot032) + \frac{1}{5}(\cdot00128) - \&c.$$

The formula $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, may also be replaced by $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$,

which is more convenient when π is to be calculated to a very large number of figures.

148. The following series are useful in many applications of Analysis.

Having given $\sin\beta = m\sin(\alpha+\beta)$, to express β in a series ascending by powers of m.

Replacing the sines by their exponential expressions, we get

$$\frac{1}{2\sqrt{-1}}\left(e^{\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}}\right) = \frac{m}{2\sqrt{-1}}\left\{e^{(\alpha+\beta)\sqrt{-1}} - e^{-(\alpha+\beta)\sqrt{-1}}\right\},$$

$$\therefore e^{2\beta\sqrt{-1}} - 1 = m\left\{e^{(\alpha+2\beta)\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right\},$$

$$\text{or } e^{2\beta\sqrt{-1}}(1 - m e^{\alpha\sqrt{-1}}) = 1 - m e^{-\alpha\sqrt{-1}},$$

$$\therefore e^{2\beta\sqrt{-1}} = \frac{1 - m e^{-\alpha\sqrt{-1}}}{1 - m e^{\alpha\sqrt{-1}}},$$

$$\therefore 2\beta\sqrt{-1} = \log(1 - m e^{-\alpha\sqrt{-1}}) - \log(1 - m e^{\alpha\sqrt{-1}})$$

$$= m\left(e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right) + \frac{m^2}{2}\left(e^{2\alpha\sqrt{-1}} - e^{-2\alpha\sqrt{-1}}\right)$$

$$+ \frac{m^3}{3}\left(e^{3\alpha\sqrt{-1}} - e^{-3\alpha\sqrt{-1}}\right) + \&c.;$$

therefore, dividing both sides by $2\sqrt{-1}$ and replacing the exponential expressions by the corresponding sines, we have

$$\beta = m \sin\alpha + \frac{m^2}{2}\sin 2\alpha + \frac{m^3}{3}\sin 3\alpha + \&c.$$

Hence if m be less than 1, β may be readily calculated; and if β be very small, then the number of seconds contained in it, will equal $\beta \div \sin 1''$, (Art. 62).

If we consider α and β to be the angles of a triangle respectively opposite to the sides a and b, and $m = \frac{b}{c}$, the above series gives an expression for an angle in terms of another angle, and the two sides containing the latter; for $\frac{\sin\beta}{\sin(\alpha+\beta)} = \frac{b}{c}$ expresses the relation between two sides and the included angle and another angle; and as this relation may be equally expressed by $\sin\alpha\cot\beta + \cos\alpha = \frac{c}{b}$,

$$\text{or } \tan\beta = \frac{b\sin\alpha}{c - b\cos\alpha} = \frac{m\sin\alpha}{1 - m\cos\alpha},$$

$$\text{or by } \tan\left(\beta + \frac{\alpha}{2}\right) = \frac{c+b}{c-b}\tan\frac{\alpha}{2} = \frac{1+m}{1-m}\tan\frac{\alpha}{2}, \text{(Art. 107.)}$$

the series may be also regarded as arising from the development of either of these latter equations.

149. We may make another application of the exponential values of the sine and cosine to express $(2\cos\theta)^n$ in terms of the cosines or sines of multiples of θ; when n is fractional.

Let ρ represent the arithmetical value of $(2\cos\theta)^n$, then when $2\cos\theta$ is positive,

$$1^n \cdot \rho = (2\cos\theta)^n = 1^n \cdot (e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})^n,$$

or $1^n.\rho = e^{2nr\pi\sqrt{-1}}\left(e^{n\theta\sqrt{-1}} + n\,e^{(n-2)\theta\sqrt{-1}} + \frac{n(n-1)}{1.2}e^{(n-4)\theta\sqrt{-1}} + \&c.\right)$;

or (Art. 144)

$$(\cos 2nr\pi + \sqrt{-1}\sin 2nr\pi)\,\rho =$$

$$\cos(2nr\pi + n\theta) + n.\cos\{2nr\pi + (n-2)\theta\}$$

$$+\frac{n(n-1)}{1.2}\cos\{2nr\pi + (n-4)\theta\} + \&c.$$

$$+\sqrt{-1}\,[\sin(2nr\pi + n\theta) + n.\sin\{2nr\pi + (n-2)\theta\}$$

$$+\frac{n(n-1)}{1.2}\sin(2nr\pi + (n-4)\theta\} + \&c.]$$

If $2\cos\theta$ be negative, so that $(2\cos\theta)^n = (-1)^n\rho$, the result will be the same, except that everywhere we must substitute $(2r+1)\pi$ instead of $2r\pi$.

The series for a fractional power of $2\sin\theta$ may be obtained in a similar manner.

Instances of the utility of Trigonometrical Formulæ.

150. In the employment of Trigonometrical formulæ, great use is made of subsidiary angles, that is, of angles whose sines, cosines, &c. do not appear in the original formula, but which are introduced to facilitate computation. We have already seen instances of this, especially where a side of a triangle is to be determined from two sides and the included angle. Other cases of frequent occurrence are the following.

(1) To adapt $a \pm b$ to logarithmic computation, where a and b are not supposed to be numbers, but expressions formed of the sines and cosines of angles. Let a be the greater of the two, then

$$a \pm b = a\left(1 \pm \frac{b}{a}\right) = a\sec^2\theta \text{ or } a\cos^2\theta,$$

assuming, for the upper and lower sign respectively,

$$\frac{b}{a} = \tan^2\theta, \quad \frac{b}{a} = \sin^2\theta.$$

(2) To convert the expression $a\cos\alpha \pm b\sin\alpha$ into another of the form $c\sin(\alpha \pm \theta)$, by means of a subsidiary angle θ.

Let $\frac{a}{b} = \tan\theta$, then

$$a\cos\alpha \pm b\sin\alpha = b(\tan\theta\cos\alpha \pm \sin\alpha) = \frac{b}{\cos\theta}\sin(\theta \pm \alpha),$$

$$\text{or } = \sqrt{a^2 + b^2}\sin(\theta \pm \alpha).$$

Other instances of the use of subsidiary angles are supplied in the next Article.

151. To solve an equation of the second degree by the aid of Trigonometrical Tables.

Let the equation be reduced to one of the forms

$$x^2 + 2px + q = 0, \text{ or } x^2 + 2px - q = 0,$$

p being positive or negative, and q essentially positive; this can always be effected.

(1) The equation

$$x^2 + 2px + q = 0 \text{ gives } x = -p \pm \sqrt{p^2 - q}.$$

If $p^2 > q$, find θ from the equation

$$\sin\theta = \sqrt{q} \div p,$$

$$\therefore\ x = -p(1 \mp \cos\theta) = -2p\sin^2\frac{\theta}{2}, \text{ or } -2p\cos^2\frac{\theta}{2},$$

which are the roots required.

If $p^2 < q$, so that the roots are impossible, make

$$\sec\theta = \sqrt{q} \div p,$$

$$\therefore\ x = -p(1 \mp \sqrt{-1}\tan\theta), \text{ which are the two roots.}$$

(2) The equation

$$x^2 + 2px - q = 0 \text{ gives } x = -p \pm \sqrt{p^2 + q};$$

find θ from the equation

$$\tan\theta = \sqrt{q} \div p,$$

$$\therefore\ x = -p(1 \mp \sec\theta) = -p\cdot\frac{\cos\theta \mp 1}{\cos\theta} = -\sqrt{q}\,\frac{\cos\theta \mp 1}{\sin\theta}$$

$$= \sqrt{q}\tan\frac{\theta}{2}, \text{ or } -\sqrt{q}\cot\frac{\theta}{2}, \text{ (Art. 49)},$$

which are the roots.

In both cases, if p be negative, we must solve the equation with p positive, and change the signs of the resulting roots. (Theory of Equations, Art. 24.)

152. Not only quadratic but cubic equations may be solved by help of Trigonometrical Tables. For since it is proved (Art. 44) that the values of z in the equation

$$z^3 - \frac{3}{4}z - \frac{a}{4} = 0 \text{ are } \cos\frac{\alpha}{3} \text{ and } \cos\frac{2\pi \pm \alpha}{3}$$

where α is the least angle whose cosine is equal to a; if we have the equation

$$x^3 - qx - r = 0,$$

and suppose $x = \frac{z}{m}$, we get

$$z^3 - m^2qz - m^3r = 0;$$

which compared with the above gives

$$m^2q = \frac{3}{4}, \quad m^3r = \frac{a}{4},$$

$$\therefore\ m = \frac{1}{2}\sqrt{\frac{3}{q}}, \quad a = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}},$$

and the values of x are

$$2\sqrt{\frac{q}{3}}\cos\frac{\alpha}{3}, \quad 2\sqrt{\frac{q}{3}}\cos\frac{2\pi \pm \alpha}{3}, \text{ where } \cos\alpha = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}.$$

Since $\cos\alpha < 1$, we must have $\frac{r^2}{4} < \frac{q^3}{27}$.

153. To find the sums of the sines and cosines of a series of angles in arithmetical progression.

We have

$$\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right) = 2\sin\frac{\beta}{2}\sin\alpha,$$

and replacing α by the values $\alpha+\beta$, $\alpha+2\beta$, &c. successively, we find

$$\cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right) = 2\sin\frac{\beta}{2}\sin(\alpha+\beta)$$

$$\cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right) = 2\sin\frac{\beta}{2}\sin(\alpha+2\beta)$$

$$\cos\left(\alpha + \frac{2n-3}{2}\beta\right) - \cos\left(\alpha + \frac{2n-1}{2}\beta\right) = 2\sin\frac{\beta}{2}\sin\{\alpha+(n-1)\beta\};$$

therefore, by addition,

$$2\sin\frac{\beta}{2}\left[\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\&c.\right.$$

$$\left.+\sin\{\alpha+(n-1)\beta\}\right]=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$=2\sin\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2},$$

$$\therefore S=\frac{\sin\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Similarly, beginning with the formula

$$\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)=2\sin\frac{\beta}{2}\cos\alpha,$$

we may find the sum of the series

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\&c. \text{ to } n \text{ terms.}$$

154. These series might also have been summed by replacing the sines and cosines by their exponential values, as in the following instance.

To find the sum S of the infinite series

$$x\sin\alpha+x^2\sin 2\alpha+x^3\sin 3\alpha+\&c.$$

Let

$$2\sqrt{-1}\sin\alpha=z-\frac{1}{z},\text{ then } 2\sqrt{-1}\sin 2\alpha=z^2-\frac{1}{z^2},\ \&c.,\text{ and } 2\cos\alpha=z+\frac{1}{z};$$

$$\therefore 2\sqrt{-1}\,S=x\left(z-\frac{1}{z}\right)+x^2\left(z^2-\frac{1}{z^2}\right)+\&c.$$

$$=xz+x^2z^2+x^3z^3+\&c.-\left(\frac{x}{z}+\frac{x^2}{z^2}+\frac{x^3}{z^3}+\&c.\right)$$

$$=\frac{xz}{1-xz}-\frac{\frac{x}{z}}{1-\frac{x}{z}}=\frac{xz-\frac{x}{z}}{1-x\left(z+\frac{1}{z}\right)+x^2}=\frac{2x\sqrt{-1}\sin\alpha}{1-2x\cos\alpha+x^2},$$

$$\therefore S=\frac{x\sin\alpha}{1-2x\cos\alpha+x^2}.$$

Similarly, for the infinite series $x\cos\alpha+x^2\cos 2\alpha+\&c.$,

$$S=\tfrac{1}{2}\cdot\left(\frac{1-x^2}{1-2x\cos\alpha+x^2}-1\right).$$

155. To resolve $\sin\theta$ and $\cos\theta$ into their factors.

Since sin θ becomes zero when $\theta = 0, \pm\pi, \pm 2\pi$, &c., $\pm n\pi$, where n is any integer whatever, and for no other values, sin θ must be divisible by,

$$\theta,\ 1 - \frac{\theta}{\pi},\ 1 + \frac{\theta}{\pi},\ 1 - \frac{\theta}{2\pi},\ \&c.,$$

and must therefore be equal to the product of all these factors multiplied by some quantity a independent of θ; hence we may assume

$$\sin\theta = a\,\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\dots$$

But limit of sin $\theta \div \theta$, when $\theta = 0$, is 1, therefore $a = 1$, and consequently

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\dots$$

Again, cos θ is reduced to zero by the values of $\theta \pm \frac{1}{2}\pi, \pm\frac{3}{2}\pi$, &c., $\pm\frac{1}{2}n\pi$, where n is any odd integer, and by no other values; therefore cos θ is divisible by $1 - \frac{2\theta}{\pi},\ 1 + \frac{2\theta}{\pi},\ 1 - \frac{2\theta}{3\pi}$, &c., and must consequently be equal to the product of all these factors multiplied by some quantity independent of θ; but since cos $\theta = 1$ when $\theta = 0$, that multiplier cannot be other than unity, and therefore

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\dots$$

156. If we put $\theta = \frac{1}{2}\pi$ in the resolution of sin θ, we get

$$1 = \tfrac{1}{2}\pi\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\dots\left\{1 - \frac{1}{(2n)^2}\right\},\ (n = \infty);$$

$$\therefore\ \tfrac{1}{2}\pi = \frac{2^2}{1\,.\,3}\cdot\frac{4^2}{3\,.\,5}\cdot\frac{6^2}{5\,.\,7}\dots\frac{(2n)^2}{(2n-1)(2n+1)},\ (n = \infty),$$

which is Wallis's formula.

Also, if by multiplying together the factors, we develope these values of sin θ and cos θ according to ascending powers of θ, and compare the results with the values given in Art. 137, for sin θ and cos θ, we find

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \frac{\pi^2}{6},$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. = \frac{\pi^2}{8}.$$

APPENDIX.

ON LOGARITHMS.

Theory of Logarithms.

1. THE values of x in the equation $n = a^x$, a being any positive number whatever, are called the logarithms of the corresponding values of n; and a is called the base of these logarithms. Hence we have this definition:

The logarithm of a number to a given base is the quantity expressing the power to which the base must be raised to become equal to the number.

The logarithm of n to the base a is written $\log_a n$; so that

$$n = a^x, \text{ and } x = \log_a n,$$

express the same relation between the three quantities n, a, x.

2. All positive numbers may be produced by the powers of any proposed positive number different from unity.

For in the equation $n = a^x$, supposing a to be a positive number greater than 1, if x assume successively and continuously all possible values from zero to infinity, it is manifest that n will receive all values from 1 to infinity. Also, if x become negative and $= -z$, then $n = a^{-z} = \frac{1}{a^z}$; and if z assume all possible values from zero to infinity, n will receive all values from 1 to zero. Hence as x changes continuously from $-\infty$ to $+\infty$, a^x changes continuously from zero to infinity, or produces all positive numbers.

If a be less than 1, writing the equation under the form $\frac{1}{n} = \left(\frac{1}{a}\right)^x$, it appears from what has been proved, since $\frac{1}{a} > 1$, that by varying x, $\frac{1}{n}$, and therefore n, will assume all positive values.

3. We cannot take for the base, unity, nor a negative quantity; for all powers of unity are unity; and the powers of

a negative quantity may be positive, or negative, or imaginary, and consequently are not subject to the law of continuity, and do not reproduce all numbers.

Since the powers of a positive quantity are always positive, negative numbers have no arithmetical logarithms.

4. Since

$$a = a^1, \quad 1 = a^0, \quad 0 = a^{-\infty} \text{ or } = a^{+\infty},$$

according as $a >$ or < 1, we deduce, from the definition of logarithms, that the logarithm of the base itself is unity, the logarithm of unity is zero, and the logarithm of zero is negative or positive infinity according as the base $>$ or < 1;

$$\text{i.e. } \log_a a = 1, \quad \log_a 1 = 0, \quad \log_a 0 = \mp \infty.$$

When the base is greater than unity, the equation $n = a^x$ shews that as the number increases, its logarithm increases; and that the logarithms of all numbers greater than 1 are positive, and the logarithms of all numbers less than 1, negative. When the base is less than 1, the contrary takes place; i.e. the logarithms of numbers greater than 1 are negative, and those of numbers less than 1 are positive.

We shall at present assume that, a being any positive quantity, if in the equation $n = a^x$, n be given, the corresponding arithmetical value of x can be found as approximately as we please.

5. If, a remaining the same, n assume successive integral values beginning from 1, and the corresponding values of x in the equation $n = a^x$ be computed and registered in order, we obtain a system of logarithms to the base a. Since a may be any positive quantity, there can be an infinite number of systems of logarithms; if however the logarithms of all numbers in any one system be known, those corresponding to any other given base may be deduced by multiplying the former by a constant multiplier, as we shall now prove.

6. A system of logarithms to base a may be transformed into a system to base b, by multiplying them by the factor $\dfrac{1}{\log_a b}$.

Let x and y denote the logarithms of any number n in two systems whose bases are a and b;

$$\therefore\ n = a^x,\ n = b^y,$$

$$\therefore\ a^x = b^y, \text{ and } a^{\frac{x}{y}} = b,$$

$$\therefore\ \frac{x}{y} = \log_a b,$$

$$\text{or } y = \frac{\ }{\log_a b} \cdot x\,;$$

that is, if we multiply x, the logarithm of any number n to the base a, by $\frac{1}{\log_a b}$, we obtain the value of y, the logarithm of the same number in the system whose base is b. This common multiplier, $1 \div \log_a b$, is called the Modulus of the system base b, relative to the system base a. Also if $x = 1$, then $n = a$, and $y = \log_b a$, therefore $\log_b a, \times \log_a b = 1$.

7. We shall now demonstrate the properties of logarithms; the fundamental one is the following:

The logarithm of a product is equal to the sum of the logarithms of its factors.

Let x, y, be the logarithms of two numbers m, n, to any base a; then

$$m = a^x,\ \ n = a^y;$$

therefore, multiplying,

$$mn = a^{x+y};$$

therefore, by definition,

$$x + y = \log_a(mn),$$

or, since $x = \log_a m$, $y = \log_a n$,

$$\log_a(mn) = \log_a m + \log_a n;$$

this being true whatever be the values of m and n, change n into np,

$$\therefore\ \log_a(mnp) = \log_a m + \log_a(np) = \log_a m + \log_a n + \log_a p;$$

and as this process may be continued to any number of factors, we conclude, generally, that the logarithm of a product is equal to the sum of the logarithms of its factors.

8. The logarithm of a quotient is equal to the logarithm of the dividend diminished by that of the divisor.

Since $m = a^x$, $n = a^y$,

$$\therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

$$\therefore \log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n.$$

9. The logarithm of any power of a number is equal to the product of the logarithm of the number by the index of the power.

Since $m = a^x$,

$$\therefore m^r = (a^x)^r = a^{rx},$$

$$\therefore \log_a(m^r) = rx = r\log_a m.$$

where r is any number whole or fractional, positive or negative.

10. The logarithm of the root of any number is equal to the logarithm of the number divided by the index of the root.

Since $m = a^x$,

$$\therefore \sqrt[r]{m} = \sqrt[r]{a^x} = a^{\frac{x}{r}},$$

$$\therefore \log_a(\sqrt[r]{m}) = \frac{x}{r} = \frac{1}{r}(\log_a m).$$

11. Hence, if we have to multiply two numbers together, or to divide one by the other, taking their logarithms out of the tables and adding or subtracting them, and then finding in the tables the number whose logarithm is equal to this sum or difference, we have the product or quotient of the numbers.

Or, if we have to find any power or root of a number, taking its logarithm out of the tables and multiplying or dividing it by the index of the power or root, and then finding in the tables the number whose logarithm is equal to this product or quotient, we have the power or root of the proposed number.

So that, by the aid of a Table of Logarithms, the arithmetical operations of multiplication and division of all numbers within the limits of the tables, may be replaced by addition and subtraction; and those of involution and evolution by the single operation of multiplying or dividing by the index; these ad-

vantages of logarithms in effecting numerical calculations, are especially seen in the case of large numbers.

12. Although there is no limit to the number of different systems of logarithms which may be formed, there are only two, of which use is ever made, the Napierian, and the Common System.

Napierian logarithms, which are always employed in Analytical investigations, have for base an incommensurable number e, whose first eight digits are 2·7182818.

Common logarithms, which are no less constantly employed in numerical calculations, have for base the number 10; and on account of the base being the same as the radix of the common scale of notation, this system has the advantage that its tables are far more comprehensive than are tables of the same size in any other system.

The multiplier by which common logarithms are formed from Napierian logarithms, or the Modulus of the Common System, is (Art. 6)

$$\mu = \frac{\quad}{\log_e 10} \quad \frac{\quad}{2{\cdot}30258509} = 0{\cdot}43429448,$$

$$\text{so that } \log_{10} n = (0{\cdot}43429448) \log_e n.$$

In the course of this Treatise we use, in algebraical analysis, $\log n$ to express $\log_e n$; and in formulæ designed for numerical calculation, we use $\log n$ to express $\log_{10} n$, any exception being specially mentioned. The suppression of the base in these two distinct cases can never lead to confusion, *Napierian* logarithms, as has been stated, being invariably employed in the former case, and common logarithms in the latter. Common logarithms are the only ones which are extensively registered in tables, and we shall now proceed to explain the arrangement and mode of using these tables.

Properties of Logarithms to base 10.

13. In the common system, as the logarithms of all numbers which are not exact powers of ten are incommensurable, their values can only be obtained approximately, and are expressed by decimals. A logarithm, therefore, will usually consist of a whole number, followed by a decimal part less than 1;

and if the number to which it corresponds be less than 1, its entire value, both decimal part and integral part, will be negative. In the tables, however, it is usual to print positive decimals only; we proceed to shew how, nevertheless, the logarithms of all numbers whatever, both the logarithms of numbers less than 1, and whose values consequently are negative, and the logarithms of numbers greater than 10 and whose values consequently have an integral as well as decimal part, can be included in a table where only positive decimals are printed.

The integral part of a logarithm is called its Characteristic.

14. A negative logarithm may be expressed so that its characteristic only shall be negative.

Let $-(n+d)$ be such a logarithm, n being its characteristic and d its decimal part; then

$$-(n+d) = -(n+1) + 1 - d = -(n+1) + d',$$

where the new decimal part d' is positive. Hence we see that the transformation of a logarithm entirely negative, into one whose characteristic only is negative, is effected by increasing the characteristic by unity, and substituting for the decimal part its defect from 1, which is readily formed by subtracting the last digit on the right from 10, and the rest from 9. And conversely, a logarithm whose characteristic only is negative may be made entirely negative, by performing on the decimal part the operation just described, and diminishing the characteristic by 1. Thus

$$\begin{aligned} -(2{\cdot}2346899) &= -3 + (1 - 0{\cdot}2346899) = -3 + {\cdot}7653101 \\ &= \bar{3}{\cdot}7653101, \end{aligned}$$

where we place the sign $-$ above the characteristic to shew that it affects only the characteristic; and the whole is used as an abbreviated mode of writing $-3 + 0{\cdot}7653101$. If, again, we wish to convert this into an expression entirely negative, we have

$$\begin{aligned} \bar{3}{\cdot}7653101 &= -3 + {\cdot}7653101 = -2 - (1 - {\cdot}7653101) \\ &= -(2{\cdot}2346899). \end{aligned}$$

Having thus shewn that the logarithm of every number whatever may be expressed so as to have a positive decimal

part; we proceed in the next place to explain how the characteristic, positive or negative, may be determined.

15. In the common system, the characteristic of the logarithm of every number having n digits in its integral part, is $n-1$; and that of every decimal having n cyphers between the decimal point and first significant digit, is negative, and equal to $n+1$.

For if a number consist of n digits, it lies between 10^{n-1} and 10^n (10^{n-1} being the least number which has n digits), and therefore its logarithm lies between the logarithms of 10^{n-1} and 10^n, or between $n-1$ and n, and therefore is composed of $n-1$ units increased by a decimal part less than 1; that is, the characteristic is equal to $n-1$.

Also, if a decimal have n cyphers between the decimal point and first significant digit, it lies between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$ (since $\frac{1}{10^{n+1}}$ is the least decimal that has n cyphers between the decimal point and first significant digit), and therefore its logarithm lies between the logarithms of these quantities, or between $-n$ and $-(n+1)$, and therefore, when expressed with a positive decimal part, is equal to $-(n+1)$ increased by a decimal part less than 1; that is, the characteristic is negative and equal to $n+1$; or it is equal to the number marking the order of the first significant digit after the decimal point.

16. In practice, a negative logarithm is always expressed so that its characteristic only is negative; and in future we shall always suppose this to be the case. When negative logarithms are expressed in this manner, there are certain peculiarities in multiplying or dividing them by any quantity, which must be attended to, and will be understood from the following instances.

If a logarithm, whose characteristic only is negative, is to be multiplied by any quantity, we must pay attention to the opposite signs of the characteristic and decimal part; thus

$$\bar{3}\cdot7653101 \times 4 = -3 \times 4 + 0\cdot7653101 \times 4 = \bar{9}\cdot0612404.$$

If a logarithm, whose characteristic only is negative, is to

be divided by any number, we must make the characteristic divisible by the number, and correct the expression by a corresponding addition. Thus, if $\bar{7}{\cdot}3295642$ is to be divided by 3, we have

$$\bar{7}{\cdot}3295642 = -9 + 2{\cdot}3295642,$$

$$\therefore \text{ the quotient is } \bar{3}{\cdot}7765214.$$

Ex. To find the 540th root of ·00007; having given

$$\log 7 = {\cdot}8450980, \quad \log 9{\cdot}824394 = {\cdot}9923057.$$

$$\log ({\cdot}00007)^{\frac{1}{540}} = \frac{1}{540}\left\{-5 + {\cdot}845098\right\} = \frac{1}{540}\left\{-540 + 535{\cdot}845098\right\}$$

$$= -1 + {\cdot}9923057 = \log ({\cdot}9824394);$$

therefore the required root is ·9824394.

17. In the common system, the same decimal part serves for the logarithms of all numbers which differ from one another only in the position of the place of units relative to the significant digits.

Let N be any whole number, having n for the characteristic and d for the decimal part of its logarithm, so that $\log N = n + d$.

First, let $N \times 10^m$ be a whole number having the same significant digits as N, but having its place of units removed m places to the right; then

$$\log (N \times 10^m) = \log 10^m + \log N = (m + n) + d,$$

which has $m + n$ for its characteristic, and the same decimal part as $\log N$.

Next, let $N \div 10^m$ be a decimal having the same significant digits as N, but having its place of units removed m places to the left; then

$$\log (N \div 10^m) = \log 10^{-m} + \log N = -m + n + d;$$

now if the proposed decimal be greater than 1, or $N > 10^m$, and therefore $\log N > m$, or n not less than m,

$$\log (N \div 10^m) = (n - m) + d;$$

which has the positive characteristic $n - m$, and the same decimal part as $\log N$;

but if the proposed decimal be less than 1, or $N < 10^m$, and therefore $\log N < m$, or n less than m, then

$$\log (N \div 10^m) = -(m - n) + d;$$

so that in this case also, provided the logarithm be expressed so that the characteristic only is negative, the decimal part is the same as that of $\log N$.

18. From the preceding observations we collect the principal advantages of tables of logarithms calculated to a base the same as the base of the system of notation, to be these; that since the characteristics of all numbers whole or fractional are known by inspection, they need not be recorded; and, the characteristic being omitted, the same record in the tables will serve for all numbers which have the same succession of significant digits, and differ only in the position of the place of units relative to those digits. Thus the record ·5386617, which in reality expresses the logarithm of 3·4567, can be made to express the logarithms of

345670, 34567, 3456·7, 345·67, 34·567, 3·4567, ·34567, ·034567,

or of any number formed by adding cyphers to the end of the former, or to the beginning of the latter immediately after the decimal point.

The same abbreviations would not be practicable in a system whose base is different from the base of the system of notation.

Thus, in the *Napierian* system,

$$\log_e 2 = ·6931471$$

$$\log_e 20 = \log_e 2 + \log_e 10 = ·6931471 + 2·3025850$$
$$= 2·9957322$$

$$\log_e ·2 = -(1·6094379)$$

so that $\log_e 2$, $\log_e 20$, $\log_e ·2$, would all require separate records; and there would be no simpler mode of expressing the logarithm of a decimal, than by converting it into a fraction and taking the difference of the logarithms of its numerator and denominator.

Mode of using Tables of Logarithms.

19. Tables of Logarithms contain all whole numbers, from 1 up to a certain limit, with their logarithms, in which the characteristic is usually suppressed. The tables in common

use contain the logarithms of all numbers from 1 up to 100000, that is, of all numbers consisting of not more than five digits, calculated to seven places of decimals.

In making use of Tables of Logarithms to effect numerical calculations, the two main problems which arise are (1) Any number being assigned, to find by means of the Tables, its logarithm, and (2) A logarithm being given, to find the corresponding number.

20. Any number whatever N being given, to find, with the aid of the tables, its logarithm.

The characteristic, whether N be a whole number or a decimal, is known by inspection.

When N, leaving the decimal point out of consideration, has not more than five digits, the decimal part of its logarithm may be taken at once out of the tables.

When N leaving the decimal point out of consideration, exceeds the limits of the tables, i. e. has more than five digits, we must transpose the decimal point, so that the integral part may be contained in the tables, and may be the greatest possible. Let n be this integral part, and d the remaining decimal part of the proposed number; l the decimal part of the logarithm of n, and δ the difference between the logarithms of the numbers n and $n+1$ (this difference is always set down in the tables), making the proportion

$$1 : d :: \delta : x,$$

the value of x, viz. $d \times \delta$, is what must be added to l to get the decimal part of the logarithm of $n + d$, which is also the decimal part of the logarithm of the proposed number.

The reason of this operation may be collected from simple considerations. If to equal differences between the numbers, corresponded equal differences between the logarithms, the differences of the logarithms would be always proportional to the differences of the numbers, and the above process would be rigorously exact. Now the inspection of the tables (see p. 119) shews us that the difference of the logarithms of consecutive whole numbers is very nearly constant, when the numbers are sufficiently large, and becomes more and more so, the larger the numbers are. Hence we are led to use the proportion in-

dicated above, and at the same time we see that the number n should be the greatest possible.

Ex. 1. Let the number be 34567.

Look for the number 3456 in the column marked N (see p. 119), then in the adjoining column marked 0, we find the first three digits 538, and carrying the eye horizontally to the column marked 7, we find the remaining four digits 6617; therefore, introducing the characteristic, $\log 34567 = 4{\cdot}5386617$.

Ex. 2. Let the number be 3456789.

Taking only five digits in the integral part,

$$\text{let } n + d = 34567{\cdot}89,$$

then considering only the integral part n, we find, as above, leaving out the characteristic,

$$\log 34567 = {\cdot}5386617\ ;$$

but $n + d$ exceeds 34567 by ·89, therefore its logarithm will exceed the above by a certain quantity to be calculated. Now, log 34568 exceeds log 34567 by ·0000126, or by 126 decimal units of the seventh order; therefore, admitting the principle that the increment of the logarithm is proportional to the increment of the number, we determine the quantity x to be added to the latter logarithm by the proportion

$$1 : 0{\cdot}89 :: 126 : x;$$

$$\therefore\ x = 126 \times {\cdot}89 = 12{\cdot}6 \times 8 + 1{\cdot}26 \times 9 = 100{\cdot}8 + 11{\cdot}34 = 112{\cdot}14,$$

or, omitting ·14 which is less than the decimal unit of the seventh order, we have $x = 112$;

$$\therefore\ \log 34567{\cdot}89 = {\cdot}5386617 + {\cdot}0000112 = {\cdot}5386729,$$

and consequently $\log 3456789 = 6.5386729$.

The trouble of multiplying 126 by 0·89 may be avoided; for in the table, below 126, we see a column of proportional parts, which contains the products of this difference by $\frac{1}{10}$, $\frac{2}{10}$, &c., or of 12·6 by 1, 2, 3, ... 9 (the last digit in the integral part of each being always increased by 1 when the decimal part,

which is neglected, is not less than ·5); whence we obtain immediately, taking the numbers opposite the digits 8 and 9,

$$126 \times 0{\cdot}8 \doteq 101, \quad 126 \times 0{\cdot}9 = 113, \text{ and } \therefore\ 126 \times {\cdot}09 = 11;$$

$$\therefore\ 126 \times 0{\cdot}89 = 101 + 11 = 112.$$

These calculations may be arranged as follows:

$N = 3456789$			
log	34567	(diff. for 1 126)	·5386617
for	·8	(12·6 × 8)	101
for	·09	(1·26 × 9)	11
∴ log	3456789		6·5386729.

The proportion between the increments of numbers, and those of their logarithms, as has been stated, is not rigorously exact; but it furnishes a sufficient approximation when the numbers are large, and their increments small. It is on that account essential to separate on the right of the given number, the smallest number possible of digits consistent with the size of the tables. Whenever $n > 10000$ and $d < 1$, it will be shewn below that the error does not affect the first seven decimals of the required logarithm.

Ex. 3. Let the number be 345678987.

Here it is necessary to separate four digits on the right, and on calculating the difference corresponding to 0·8987, (which may be done by taking the proportional part for each digit out of the table) we find

$$126 \times 0{\cdot}8 = 101, \quad 126 \times {\cdot}09 = 11{\cdot}3, \quad 126 \times {\cdot}008 = 1{\cdot}01,$$

$$126 \times {\cdot}0007 = {\cdot}088.$$

Hence, on adding these partial products together, we find the required log = 8·5386730; and we see that the last product, viz. that which results from the digit 7, has no influence on the seventh decimal of the logarithm. This example shews that, in general, when it is necessary to separate on the right more than three digits, we may count the fourth digit and all the others as zero.

When N contains decimals, we must leave the decimal point out of consideration, and proceed exactly as if N were an

integer; the decimal part of the logarithm, as has been proved (Art. 17), will not be altered, and the characteristic, which is known beforehand, may then be prefixed.

Thus,

$$\log 34{\cdot}56789 = 1{\cdot}5386729, \quad \log {\cdot}003456789 = \bar{3}.5386729.$$

21. Hence to find the logarithm of a number which consists of more than five digits, and which is therefore beyond the limits of the ordinary tables, we have the following rule:

Leaving the decimal point out of consideration (if the number be a decimal), find the decimal part of the logarithm of the number formed by the first five digits; next in the column of differences have recourse to the difference immediately above the logarithm just mentioned, and from the column of proportional parts take the number corresponding to the sixth, 1-10th of the number corresponding to the seventh, and 1-100th of the number corresponding to the eighth digit, if there be any such digits; add these numbers in their proper places (recollecting that the unit's place of the first is a decimal of the seventh order) to the logarithm of the number formed by the first five digits, and the result is the required decimal part of the logarithm of the number of 6, 7, or 8 places; to which the characteristic, known by inspection, may then be prefixed.

22. The second problem which occurs in using the tables is, a logarithm L being given, to find, with the aid of the tables, the corresponding number.

When the decimal part, l, of L is positive, and found exactly in the tables, we have only to take out the corresponding number; and the given characteristic will determine the position of the place of units.

When the decimal part, l, of the given logarithm is not found exactly in the tables, let n be the integer whose logarithm has a decimal part l' immediately inferior to l, δ the difference between the decimal part l' and that which immediately follows it in the tables corresponding to $n+1$, and δ' the difference $l-l'$; making the proportion

$$\delta : \delta' :: 1 : x,$$

we find the fourth term $x = \delta' \div \delta$, which we must reduce to a

decimal, and add it to the number n; we thus find a number whose logarithm has for decimal part l, and from which we can deduce the required number, having regard to the characteristic, as in the preceding case.

When the given logarithm is entirely negative, we must transform it so as to have the decimal part positive, and then proceed as explained above.

Ex. 1. Let the given logarithm be 5.5386617.

In the column marked 0 we seek the logarithm which is next less than the decimal part of L, which we find to be 5385737, opposite to the number 3456 in the column marked N (see p. 119); then advancing in the horizontal line to the column marked 7, we find the four last decimals 6617 of L. Hence affixing the digit 7 to 3456 we have 34567, and since the given characteristic is 5, the required number is 345670.

Ex. 2. Let the given logarithm be 2·5386729.

In seeking for the decimal part in the tables as above, we find that it lies between ·5386617 and ·5386743; if it were exactly equal to the former, the corresponding number would be 34567, but as it surpasses ·5386617 by 112, there will be an increase in 34567; and as the difference of the logarithms corresponding to an increase of unity in 34567 is 126, the required increment x, admitting the common principle of proportional parts, will be found by the proportion

$$126 : 112 :: 1 : x = 112 \div 126 = 0{\cdot}89.$$

Therefore the required number, neglecting the decimal point, is 3456789, and as the given characteristic is 2, the required number is 345·6789. By means of the table of proportional parts of the tabular difference, we may avoid the trouble of dividing 112 by 126. In the table of proportional parts of the difference 126, the part next less than 112 is 101, which corresponds to ·8, and there remains 11. If we put a zero to the right of 11, we have 110, which differs very little from the part 113, which has 9 opposite to it; we conclude therefore that 110 corresponds to ·9, and 11 to :09; consequently the required number is composed of the digits 3456789; and taking account of the characteristic, the number is 345·6789, as before.

These calculations may be arranged as follows:

$L = 2{\cdot}5386729$		
for	$0{\cdot}5386617$	34567
1st rem.	112	
for	101	0·8
2nd rem.	11	
for	11	0·09,
	∴ required number	345·6789

Ex. 3. Let the given logarithm be $-(1{\cdot}4614822)$.

Reducing this to another of which the characteristic only is negative, we have

$$L = \bar{2}{\cdot}5385178,$$

and proceeding as in the last example, the number sought is ·03455555.

23. Hence to find the number corresponding to a given logarithm not exactly found in the tables, we have the following rule.

Leaving the characteristic out of consideration, find the tabular logarithm immediately inferior to the given logarithm, and take out the corresponding five digits; also find the difference δ' of these logarithms; next in the column of differences have recourse to the difference which stands next above the tabular logarithm just mentioned, and in the table of proportional parts look for p that which is next less than δ', and take out the number opposite to it for the sixth digit; subtract p from δ' and look for q the proportional part next less than ten times their difference δ'', and take out the number opposite to q for the seventh digit; subtract q from δ'' and look for the proportional part next less than 10 times their difference, and take out the number opposite to it for the eighth digit. Having thus found the seven or eight digits of which the number is composed, the given characteristic will determine the position of the decimal point.

The following is a specimen of the ordinary tables of logarithms of numbers; the meaning of the cyphers printed in a smaller type is, that the first three figures of those logarithms

are found in the line next below the numbers, their fourth figures having changed from 9s to 0s.

Log. 537, N. 345.

N	0			3					8	9	Diff.
3450	5378191	8317	8443	8569	8694	8820	8946	9072	9198	9324	126
51	9450	9575	9701	9827	9953	0079	0205	0330	0456	0582	
52	5380708	0834	0959	1085	1211	1337	1463	0588	1714	1840	
53	1966	2092	2217	2343	2469	2595	2720	2846	2972	3098	
54	3223	3349	3475	3601	3726	3852	3978	4103	4229	4355	
55	4481	4606	4732	4858	4983	5109	5235	5360	5486	5612	
56	5737	5863	5989	6114	6240	6366	6491	6617	6743	6868	
57	6994	7119	7245	7371	7496	7622	7747	7873	7999	8124	
58	8250	8375	8501	8627	8752	8878	9003	9129	9255	9380	
59	9506	9631	9757	9882	0008	0133	0259	0384	0510	0635	

Pro.	126
1	13
2	25
3	38
4	50
5	63
6	76
7	88
8	101
9	113

24. In actual calculations, it is often necessary to add several logarithms together, and from their sum to subtract other logarithms. In such cases it is convenient to reduce all the operations to a single addition, by adding the arithmetical complement of every logarithm that is to be subtracted (that is, its defect from 10), and diminishing the final result by as many tens as there are complements; the complements being all formed by the uniform operation of subtracting the last digit of each from 10, and all the other digits from 9.

Ex. To find x within ·01 where $x = \dfrac{7340 \times \cdot 3549}{681 \cdot 8 \times 593 \cdot 1}$,

$$\begin{aligned} \log\ 7340 &= 3 \cdot 8656961 \\ \log\ 3549 &= 3 \cdot 5501060 \\ \text{comp. } \log 681 \cdot 8 &= 7 \cdot 1663430 \\ \text{comp. } \log 593 \cdot 1 &= 7 \cdot 2268721 \\ \hline \therefore\ \text{sum} - 20, \text{ or } \log x &= 1 \cdot 8090172 \\ x &= 64 \cdot 42. \end{aligned}$$

Exponential and Logarithmic Series.

25. To expand a^x in a series ascending by powers of x, i.e. to expand a number in a series ascending by powers of its logarithm.

Since when $x = 0$, a^x becomes 1, the first term of the development will be 1; also, since $a = 1 + (a - 1)$, and therefore

$$\{1 + (a - 1)\}^x = 1 + x(a - 1) + x(x - 1)\frac{(a - 1)^2}{1 \cdot 2}$$

$$+ x(x-1)(x-2)\frac{(a-1)^3}{1.2.3} + \&c.$$

$$= 1 + \left\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.\right\} x$$

$$+ \text{ terms involving } x^2, x^3, \&c.$$

we may assume

$$a^x = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \&c.,$$

where p_1, the coefficient of x,

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.,$$

and p_2, p_3, &c. the coefficients of the other powers of x, are indeterminate quantities, independent of x. To find them, we shall make use of the characteristic property of the expression a^x, viz. $a^x \times a^y = a^{x+y}$. Changing, in the above series, x into y, and into $x + y$ successively, we get

$$a^y = 1 + p_1 y + p_2 y^2 + p_3 y^3 + \&c.$$

$$a^{x+y} = 1 + p_1(x+y) + p_2(x+y)^2 + p_3(x+y)^3 + \&c.;$$

therefore, in order to verify the property $a^x \times a^y = a^{x+y}$, we must have

$$(1 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + \&c.)$$

$$\times (1 + p_1 y + p_2 y^2 + p_3 y^3 + p_4 y^4 + \&c.)$$

$$= 1 + p_1(x+y) + p_2(x+y)^2 + p_3(x+y)^3 + p_4(x+y)^4 + \&c.$$

If we effect the operations indicated, and consider in particular the part which involves the first power of y in each member, since these parts must be equal, we shall have

$$p_1 + p_1^2 x + p_1 p_2 x^2 + p_1 p_3 x^3 + \&c. = p_1 + 2p_2 x + 3p_3 x^2 + 4p_4 x^3 + \&c.$$

and as this equation holds for all values of x, the coefficients of like powers must be equal,

$$\therefore\ 2p_2 = p_1^2, \quad 3p_3 = p_1 p_2, \quad 4p_4 = p_1 p_3, \ \&c.$$

$$\text{or } p_2 = \frac{p_1^2}{2}, \quad p_3 = \frac{p_1^3}{1.2.3}, \quad p_4 = \frac{p_1^4}{1.2.3.4}, \quad \&c.$$

and substituting these values, the series becomes

$$a^x = 1 + \frac{p_1 x}{1} + \frac{p_1^2 x^2}{1.2} + \frac{p_1^3 x^3}{1.2.3} + \frac{p_1^4 x^4}{1.2.3.4} + \&c.,$$

where $p_1 = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \&c.$

26. The above expansion of a^x being true for all values of x, make

$$p_1 x = 1, \text{ or } x = \frac{1}{p_1};$$

$$\therefore\ a^{\frac{1}{p_1}} = 2 + \frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \&c.$$

According to the usual notation, let e represent this numerical series; $\therefore\ a^{\frac{1}{p_1}} = e$, and $a = e^{p_1}$, and consequently p_1 is the logarithm of a in a system whose base is e, or $p_1 = \log_e a$;

$$\therefore\ a^x = 1 + (\log_e a)\,\frac{x}{1} + (\log_e a)^2\frac{x^2}{1\,.\,2} + (\log_e a)^3\frac{x^3}{1\,.\,2\,.\,3} + \&c.$$

If the exponential quantity be e^x, making $a = e$ and $\log_e a = \log_e e = 1$, we get

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.;$$

and we also have

$$\log_e a = p_1 = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

27. The quantity e or $2 + \frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \&c.$ is incommensurable.

For since $\frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \&c.$ is less than the infinite geometrical progression

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \&c. \text{ the sum of which is } 1,$$

e lies between 2 and 3; and therefore if e can be expressed exactly by any number, it must be by a fraction $\frac{m}{n}$. Suppose then that

$$\frac{m}{n} = 2 + \frac{1}{2} + \frac{1}{2\,.\,3} + \ldots + \frac{1}{2\,.\,3\ldots n} + \frac{1}{2\,.\,3\ldots(n+1)} + \&c.$$

therefore, multiplying both sides by $2\,.\,3\ldots(n-1)n$, we get

$$2\,.\,3\ldots(n-1)m = N + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.,$$

where N denotes an integer. But since $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.$ is less than the geometrical progression $\frac{1}{n+1} + \frac{1}{(n+1)^2} + \&c.$, the sum of which is $\frac{1}{n}$, it would follow that by adding N to a fraction less than $\frac{1}{n}$, the result is a whole number, which is absurd. Therefore e is incommensurable.

If we convert a small number of terms of the series into decimals, we find the value of e, as stated above,

$$= 2 + \cdot 5 + \cdot 166666 + \cdot 041666 + \cdot 008333 + \cdot 001388 + \&c.,$$

$$= 2 \cdot 7182818;$$

and we may easily find a limit to the error committed in taking only n terms of this series for the value of e; for the error, which is the sum of the remaining terms

$$= \frac{1}{1.2.3\ldots n}\left\{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.\right\},$$

which, by what has been shewn, is less than $\frac{1}{1.2.3\ldots n} \cdot \frac{1}{n}$.

28. To expand $\log_a(1+x)$ in a series ascending by powers of x.

We cannot attempt to develope $\log_a x$ in a series of the form $A + Bx + Cx^2 + \&c.$, since when $x = 0$, $\log_a x$ becomes infinite; but we may seek to develope $\log_a(1+x)$ in a series of that form, since this becomes zero when $x = 0$ and therefore cannot involve any term independent of x, or any negative power of x; consequently we may assume

$$\log_a(1+x) = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.\ldots(1);$$

change x into $x+y$, and we have

$$\log_a(1+x+y) = A(x+y) + B(x+y)^2 + C(x+y)^3 + \&c.$$

Now $\quad 1 + x + y = (1+x)\left(1 + \frac{y}{1+x}\right),$

$$\therefore \log_a(1+x+y) = \log_a(1+x) + \log_a\left(1 + \frac{y}{1+x}\right);$$

but changing x into $\dfrac{y}{1+x}$ in equation (1), we have

$$\log_a\left(1+\frac{y}{1+x}\right)=\frac{Ay}{1+x}+\frac{By^2}{(1+x)^2}+\&c.,$$

$$\therefore\ \log_a(1+x+y)=\log_a(1+x)+\frac{Ay}{1+x}+\frac{By^2}{(1+x)^2}+\&c.$$

We have thus a second expression for $\log_a(1+x+y)$; and as the coefficients of similar powers of y must be equal in the two, we find

$$A+2Bx+3Cx^2+4Dx^3+\&c=\frac{A}{1+x};$$

$\therefore$ multiplying both sides by $1+x$, and transposing,

$$(A+2B)x+(2B+3C)x^2+(3C+4D)x^3+\&c.=0.$$

Therefore, since x is indeterminate, $A+2B=0$, $2B+3C=0$, $3C+4D=0$, &c; or $B=-\frac{1}{2}A$, $C=\frac{1}{3}A$, $D=-\frac{1}{4}A$, &c. and consequently

$$\log_a(1+x)=A\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.\right).$$

Now as this is true for all values of x, put $1+x=a$, then (Art. 26),

$$\log_a a=1=A\{a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\}=A\log_e a;$$

$$\therefore\ \log_a(1+x)=\frac{1}{\log_e a}(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\&c.)$$

29. The coefficient A, however, may be determined directly, without assuming the expansion of $\log_e a$. For since

$$\frac{\log_a(1+x)}{x}=A\left(1-\frac{x}{2}+\frac{x^2}{3}-\&c.\right),$$

it appears that A is the value of $\dfrac{\log_a(1+x)}{x}$ when $x=0$. But

making $x=\dfrac{1}{n}$, we have

$$\frac{\log_a(1+x)}{x}=n\log_a\left(1+\frac{1}{n}\right)=\log_a\left(1+\frac{1}{n}\right)^n.$$

Now

$$\left(1+\frac{1}{n}\right)^n = 1+n\,.\frac{1}{n}+\frac{n(n-1)}{1\,.\,2}\,.\frac{1}{n^2}+\&c.$$

$$=2+\left(\frac{1}{2}-\frac{1}{2n}\right)+\left(\frac{1}{2}-\frac{1}{2n}\right)\left(\frac{1}{3}-\frac{2}{3n}\right)+\&c.;$$

and when we make $n=\infty$, which corresponds to the supposition $x=0$, the second member is reduced to

$$2+\frac{1}{2}+\frac{1}{2\,.\,3}+\&c. \text{ or } e;$$

$$\therefore A=\log_a e=\frac{1}{\log_e a}; \text{ (Art. 6)}$$

$$\therefore \log_a(1+x)=\frac{1}{\log_e a}(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\&c.);$$

and supposing $a=e$, so that $\log_e a=\log_e e=1$,

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\&c.$$

30. We are now able to prove the proportion assumed above between the increments of numbers and those of their logarithms, when the numbers are large and their differences small. For since

$$\log_{10}(n+d)-\log_{10}n=\log_{10}\left(1+\frac{d}{n}\right)=\frac{\mu d}{n}\left(1-\frac{d}{2n}+\frac{d^2}{3n^2}-\&c.\right)=\mu\frac{d}{n},$$

if n be a large number, and d a fraction of a unit, so that all the terms except the first may be neglected; therefore, making $d=1$,

$$\log_{10}(n+1)-\log_{10}n=\frac{\mu}{n};$$

but $\log_{10}(n+1)-\log_{10}n$ is the difference of consecutive logarithms, and is given in the tables $=\delta$;

$$\therefore \log_{10}(n+d)-\log_{10}n=d\delta,$$

$$\text{or } \log_{10}(n+d)=\log_{10}n+d\delta,$$

which gives the logarithm of a number by means of those of the consecutive integers between which it lies.

And conversely, if $\log_{10}(n+d)-\log_{10}n$ be given $=\delta'$,

then $d = \frac{\delta'}{\delta}$, which is the fraction to be added to n to form the number corresponding to a logarithm lying between the logarithms of n and $n + 1$.

31. To investigate the errors committed in calculating either the logarithm of a given number, or the number corresponding to a given logarithm, by the principle of proportional parts.

The real value of $\log (n + d) - \log n$ lies between

$$\frac{\mu d}{n}\left(1 - \frac{d}{2n}\right) \text{ and } \frac{\mu d}{n},$$

and therefore, *a fortiori* between

$$\frac{\mu d}{n}\left(1 - \frac{1}{2n}\right) \text{ and } \frac{\mu d}{n}$$

because the lower limit is now made less; also its approximate value $d\delta$ plainly lies between the same inferior and superior limits; therefore, as the difference of these values must be less than the difference of the limits between which they are both contained, the error in taking $d\delta$ for the value of $\log (n+d) - \log n$ is less than $\frac{\mu d}{2n^2}$; which fraction, provided $n > 10{,}000$ and $d < 1$, is less than $0.43 \div 200{,}000{,}000$ i.e. less than a quarter of a decimal unit of the eighth order; the error, therefore, does not affect the first seven decimals of the required logarithm.

Again, the error in assuming d to be equal to $\frac{\delta'}{\delta}$ is $\frac{d\delta - \delta'}{\delta}$, which, by what precedes, is less than $\frac{\mu d}{2n^2\delta}$; but

$$\delta > \frac{\mu}{n}\left(1 - \frac{1}{2n}\right) > \mu . \frac{2n-1}{2n^2}; \therefore a\ fortiori\ d - \frac{\delta'}{\delta} < \frac{d}{2n-1} < \frac{1}{20{,}000}$$

provided $n > 10{,}000$; consequently, the error committed in taking $\frac{\delta'}{\delta}$ for the value of d, has no influence on the first four decimals of the required number.

32. The series $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c.$ is not convergent except for values of x less than 1, but it will enable us to deduce others which serve for greater values of x. Changing x into $-x$, we get

$$\log_e(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c.;$$

and subtracting $\log_e(1-x)$ from $\log_e(1+x)$, we find the logarithm of the quotient of $1+x$ divided by $1-x$, viz.

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left\{\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\}.$$

In this, replace $\frac{1+x}{1-x}$ by $1+\frac{z}{n}$, so that $x = \frac{z}{2n+z}$: then since

$$\log_e\left(1+\frac{z}{n}\right) = \log_e\left(\frac{n+z}{n}\right) = \log_e(n+z) - \log_e n,$$

we have

$$\log_e(n+z) = \log_e n$$

$$+ 2\left\{\frac{z}{2n+z} + \frac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n+z}\right)^5 + \&c.\right\},$$

a convenient formula by which, if n be a large number and z a small one, we may ascend from $\log_e n$ to $\log_e(n+z)$.

If $z = 1$, we have

$$\log_e(n+1) = \log_e n$$

$$+ 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \&c.\right\} (1).$$

33. Again, replacing $\frac{1+x}{1-x}$ by $\frac{m}{n}$, and therefore x by $\frac{m-n}{m+n}$ we have

$$\log_e\left(\frac{m}{n}\right) = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\}, (2)$$

which will give the logarithm of any number m by means of that of an inferior number n; and from which we may deduce a variety of formulæ, taking for m and n expressions capable of being resolved into simple factors, and which differ only in their last terms.

Thus, let $m = x^2$, $n = x^2 - 1$, then $\dfrac{m-n}{m+n} = \dfrac{1}{2x^2-1}$, and

$$\log_e\left(\frac{m}{n}\right) = 2\log_e x - \log_e(x-1) - \log_e(x+1),$$

$$\therefore \log_e(x+1) = 2\log_e x - \log_e(x-1),$$

$$-2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3 + \&c.\right\}, \quad (3)$$

a formula by which, having given the *Napierian* logarithms of consecutive numbers $x-1$ and x, we may find that of the number next following. Or, we may thus obtain another formula for the same purpose. Replacing x by $\dfrac{1}{x}$ in formula of Art. 32, and dividing by $2x$, we get

$$\frac{1}{2x}\log\left(\frac{x+1}{x-1}\right) = \frac{1}{x^2} + \frac{1}{3x^4} + \frac{1}{5x^6} + \&c.;$$

but $\log(x^2-1) = \log x^2\left(1 - \dfrac{1}{x^2}\right) = 2\log x - \dfrac{1}{x^2} - \dfrac{1}{2x^4} - \dfrac{1}{3x^6} - \&c;$

therefore, adding,

$$\left(1+\frac{1}{2x}\right)\log(x+1) + \left(1-\frac{1}{2x}\right)\log(x-1) = 2\log x - \frac{1}{2\,.\,3x^4}$$

$$- \frac{2}{3\,.\,5x^6} - \frac{3}{4\,.\,7x^8} - \&c.$$

Similarly, making in formula (2)

$$m = (x+2)(x-1)^2 = x^3 - 3x + 2,$$

$$n = (x-2)(x+1)^2 = x^3 - 3x - 2,$$

we obtain a formula connecting the logarithms of $x-2$, $x-1$, $x+1$, $x+2$.

34. In making use of these series to actually compute logarithms, for low primes we should use (1). Thus, making $n = 1, 2$, we have

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \&c.\right\}$$

$$= 2\{\cdot333333333 + \cdot012345678 + \cdot000823045 + \cdot000065321$$
$$+ \cdot000005645 + \cdot000000513 + \&c.\}$$

$$= 2\{\cdot346573536\} = \cdot693147172.$$

$$\log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \&c.\right\}$$

$$\log_e 2 + \frac{4}{10} + \frac{1}{3}\cdot\frac{16}{10^3} + \frac{1}{-}\cdot\frac{64}{10^5} + \&c. = 1{\cdot}09861229.$$

For high primes we should use formula (3); and for composite numbers, having resolved them into powers of their prime factors, we should find their logarithms by means of the logarithms of their prime factors. Thus

$$\log_e 5 = 2\log_e 4 - \log_e 3 - 2\left\{\frac{1}{31} + \frac{1}{3}\left(\frac{1}{31}\right)^3 + \frac{1}{5}\left(\frac{1}{31}\right)^5 + \&c.\right\}$$

$$= 2{\cdot}772588688 - 1{\cdot}09861229$$

$$-2\left\{{\cdot}032258064 + {\cdot}000011189\right\}$$

$$= 1{\cdot}6094379.$$

$$\log_e 10 = \log_e 2 + \log_e 5 = 2{\cdot}3025851.$$

The *Napierian* logarithms of all numbers being thus computed, the logarithms to base 10 are found from them, as has been stated, by multiplying them by the constant factor

$$\mu = 1 \div \log_e 10 = {\cdot}43429448.$$

Series for Logarithmic Sines and Cosines.

35. We shall here shew how, without knowing the values of the sines and cosines themselves, the values of their logarithms may be computed to any degree of exactness.

It is proved (Art. 155) that

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots$$

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

Put $\theta = \frac{m}{n}\frac{\pi}{2}$, then

$$\sin\frac{m}{n}\frac{\pi}{2} = \frac{m}{n}\frac{\pi}{2}\left(1 - \frac{m^2}{2^2n^2}\right)\left(1 - \frac{m^2}{4^2n^2}\right)\left(1 - \frac{m^2}{6^2n^2}\right)\ldots$$

$$\cos\frac{m}{n}\frac{\pi}{2} = \left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{3^2n^2}\right)\left(1 - \frac{m^2}{5^2n^2}\right)\ldots$$

Hence, taking the common logarithms of both sides of these equations, and expanding the logarithms of all the binomial factors (except the first in each, to ensure sufficient accuracy) by Art. 28, we get

$$\log \sin \frac{m}{n}\frac{\pi}{2} = \log \pi + \log m + \log (2n + m) + \log (2n - m)$$

$$- 3 (\log 2 + \log n)$$

$$- \mu \left\{ \left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \&c.\right) \frac{m^2}{n^2} + \frac{1}{2}\left(\frac{1}{4^4} + \frac{1}{6^4} + \&c.\right) \frac{m^4}{n^4} \right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{4^6} + \frac{1}{6^6} + \&c.\right) \frac{m^6}{n^6} + \&c. \right\}$$

$$\log \cos \frac{m}{n}\frac{\pi}{2} = \log (n + m) + \log (n - m) - 2 \log n$$

$$- \mu \left\{ \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.\right) \frac{m^2}{n^2} + \frac{1}{2}\left(\frac{1}{3^4} + \frac{1}{5^4} + \&c.\right) \frac{m^4}{n^4} \right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{3^6} + \frac{1}{5^6} + \&c.\right) \frac{m^6}{n^6} + \&c. \right\}$$

from which series, by giving m and n proper values, the values of the logarithms of all sines and cosines may be computed, independently of the values of the sines and cosines themselves.

PROBLEMS.

1. Express $\cos m\theta \cos n\theta \cos r\theta$ by the sum of a series of cosines of multiples of θ.

First we have $2\cos m\theta \cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta$;

$$\therefore\ 4\cos m\theta \cos n\theta \cos r\theta$$

$$= 2\cos(m+n)\theta\cos r\theta + 2\cos(m-n)\theta\cos r\theta$$

$$= \cos(m+n+r)\theta + \cos(m+n-r)\theta + \cos(m-n+r)\theta$$
$$+ \cos(m-n-r)\theta.$$

2. Prove that $\sin 3A = 4\sin A\sin(60^0 + A)\sin(60^0 - A)$.

The second member $= 4\sin A(\sin^2 60^0 - \sin^2 A)$

$$= 4\sin A\left(\frac{3}{4} - \sin^2 A\right) = 3\sin A - 4\sin^3 A.$$

3. Express $\sin^3\theta \times \cos^2\theta$ by a series of sines of multiples of θ.

$$\sin^3\theta \,.\, \cos^2\theta = \frac{1}{4}\sin\theta(\sin 2\theta)^2 = \frac{1}{8}\sin\theta(1 - \cos 4\theta)$$

$$= \frac{1}{8}\sin\theta - \frac{1}{16}(\sin 5\theta - \sin 3\theta).$$

4. In any circle the chord of an arc of 108^0 is equal to the sum of the chords of 36^0 and 60^0.

If r be the radius of the circle, chord of $108^0 = 2r\sin 54^0$

$$= \tfrac{1}{2}r(\sqrt{5}+1) = \tfrac{1}{2}r(\sqrt{5}-1) + r = 2r\sin 18^0 + 2r\sin 30^0$$
$$= \text{chord of } 36^0 + \text{chord of } 60^0.$$

5. Prove that $\tan(\alpha+\beta) = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta}$.

The 2nd member $= \dfrac{2\sin^2\alpha - 2\sin^2\beta}{\sin 2\alpha - \sin 2\beta} = \dfrac{\cos 2\beta - \cos 2\alpha}{\sin 2\alpha - \sin 2\beta}$

$$\frac{2\sin(\alpha+\beta)\sin(\alpha-\beta)}{2\cos(\alpha+\beta)\sin(\alpha-\beta)} = \tan(\alpha+\beta).$$

6. Prove that $\dfrac{1 + (\operatorname{cosec} \alpha \tan x)^2}{1 + (\operatorname{cosec} \beta \tan x)^2} = \dfrac{1 + (\cot \alpha \sin x)^2}{1 + (\cot \beta \sin x)^2}$.

The 1st member $= \dfrac{\cos^2 x + (1 + \cot^2 \alpha) \sin^2 x}{\cos^2 x + (1 + \cot^2 \beta) \sin^2 x}$

$$= \frac{1 + (\cot \alpha \sin x)^2}{1 + (\cot \beta \sin x)^2}.$$

7. Prove that $\sin^{-1} \sqrt{\dfrac{x}{a + x}} = \tan^{-1} \sqrt{\dfrac{x}{a}}$.

Let $\tan^{-1} \sqrt{\dfrac{x}{a}} = \theta$; $\therefore \tan \theta = \sqrt{\dfrac{x}{a}}$,

$$\sin \theta = \sqrt{\frac{x}{a}} \div \sqrt{1 + \frac{x}{a}} = \sqrt{\frac{x}{a + x}};$$

$$\therefore \theta = \sin^{-1} \sqrt{\frac{x}{a + x}} = \tan^{-1} \sqrt{\frac{x}{a}}.$$

8. $\tan^{-1}\left(\dfrac{x \cos \phi}{1 - x \sin \phi}\right) - \tan^{-1}\left(\dfrac{x - \sin \phi}{\cos \phi}\right) = \phi$.

For $\tan^{-1}\left(\dfrac{x - \sin \phi}{\cos \phi}\right) + \tan^{-1} (\tan \phi)$

$$= \tan^{-1} \frac{x \sec \phi}{1 - \dfrac{(x - \sin \phi) \sin \phi}{\cos^2 \phi}} = \tan^{-1}\left(\frac{x \cos \phi}{1 - x \sin \phi}\right).$$

9. $\tan^2 \alpha - \tan^2 \beta = \dfrac{\sin (\alpha + \beta) \sin (\alpha - \beta)}{\cos^2 \alpha \cos^2 \beta}$.

The 1st member $= (\tan \alpha + \tan \beta)(\tan \alpha - \tan \beta)$

$$= \frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} \, \frac{\sin (\alpha - \beta)}{\cos \alpha \cos \beta}.$$

If $\sin \theta = \sin \alpha \,.\, \sin (\phi + \theta)$ then $\tan \theta = \dfrac{\sin \alpha \sin \phi}{1 - \sin \alpha \cos \phi}$.

For $\sin \theta = \sin \alpha \sin \phi \cos \theta + \sin \alpha \cos \phi \sin \theta$;

$$\therefore \tan \theta = \sin \alpha \sin \phi + \sin \alpha \cos \phi \tan \theta,$$

$$\text{or } \tan \theta (1 - \sin \alpha \cos \phi) = \sin \alpha \sin \phi.$$

10. If $\dfrac{\tan(a-b)}{\tan a} = 1 - \dfrac{\sin^2 c}{\sin^2 a}$, then $\tan a \tan b = \tan^2 c$.

For $\tan a - \tan(a-b) = \dfrac{\sin^2 c}{\sin^2 a} \tan a$,

and $1 + \tan a \,.\, \tan(a-b) = 1 + \tan^2 a - \dfrac{\sin^2 c}{\cos^2 a} = \dfrac{\cos^2 c}{\cos^2 a}$;

$$\therefore\ \tan b = \tan^2 c \,.\, \cot a.$$

If $\cos^3\theta + a\cos\theta = b$, $\sin^3\theta + a\sin\theta = c$, then shall

$$\sin 2\theta = 2\sqrt{\frac{(1+a)^2 - b^2 - c^2}{4a+3}}.$$

Adding the equations together, and then squaring the result, we get successively

$$(\cos\theta + \sin\theta)(1 + a - \cos\theta\sin\theta) = b + c,$$

$$(1 + \sin 2\theta)(2 + 2a - \sin 2\theta)^2 = 4(b+c)^2,$$

or $4(1+a)^2 + 4(a+a^2)\sin 2\theta - (4a+3)\sin^2 2\theta + \sin^3 2\theta = 4(b+c)^2$;

but $\qquad 4(a + a^2)\sin 2\theta + \sin^3 2\theta = 8bc,$

obtained by multiplying the equations together; therefore, subtracting,

$$4(1+a)^2 - (4a+3)\sin^2 2\theta = 4(b^2 + c^2).$$

11. If $\cos 60^0 = \sin 36^0 \cos a$, $\cos 36^0 = \sin 60^0 \cos b$, then $\tan a + \tan b = 1$. The first equation gives

$$\sec a = \tfrac{1}{2}\sqrt{10 - 2\sqrt{5}},\ \therefore\ \tan a = \tfrac{1}{2}(\sqrt{5} - 1);$$

similarly, the second equation gives $\tan b = \frac{1}{2}(3 - \sqrt{5})$;

$$\therefore\ \tan a + \tan b = 1.$$

If $\tan\theta = \cos a \tan\phi$,

$$\text{then } \tan(\phi - \theta) = \frac{\tan^2 \frac{1}{2} a \sin 2\phi}{1 + \tan^2 \frac{1}{2} a \cos 2\phi}.$$

For $\tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$

$$= \frac{\tan\phi(1 - \cos a)}{1 + \tan^2\phi\cos a} = \frac{2\sin\phi\cos\phi(1 - \cos a)}{2\cos^2\phi + 2\sin^2\phi\cos a}$$

$$\frac{\sin 2\phi(1 - \cos a)}{1 + \cos a + (\cos^2\phi - \sin^2\phi)(1 - \cos a)} = \frac{\tan^2\frac{1}{2}a \sin 2\phi}{1 + \tan^2\frac{1}{2}a\cos 2\phi}.$$

12. If $m = \operatorname{cosec}\theta - \sin\theta$, $n = \sec\theta - \cos\theta$,

$$\text{then } m^{\frac{4}{3}}n^{\frac{2}{3}} + m^{\frac{2}{3}}n^{\frac{4}{3}} = 1.$$

For $m = \dfrac{\cos^2\theta}{\sin\theta}$, $n = \dfrac{\sin^2\theta}{\cos\theta}$, $\therefore\ mn = \cos\theta\sin\theta$;

and $m^{\frac{2}{3}} + n^{\frac{2}{3}} = \dfrac{\cos^{\frac{4}{3}}\theta}{\sin^{\frac{2}{3}}\theta} + \dfrac{\sin^{\frac{4}{3}}\theta}{\cos^{\frac{2}{3}}\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{(\sin\theta\cos\theta)^{\frac{2}{3}}} = \dfrac{1}{(mn)^{\frac{2}{3}}}$.

13. To find m in terms of α from the equations

$$m^2 = 3\cos^2\phi + 1, \quad \tan^3\tfrac{1}{2}\phi = \tan\alpha.$$

$$m^2 = 3\left(\frac{1-\tan^2\frac{1}{2}\phi}{1+\tan^2\frac{1}{2}\phi}\right)^2 + 1 = 3\left(\frac{1-\tan^{\frac{2}{3}}\alpha}{1+\tan^{\frac{2}{3}}\alpha}\right)^2 + 1$$

$$= \frac{4(1-\tan^{\frac{2}{3}}\alpha + \tan^{\frac{4}{3}}\alpha)}{(1+\tan^{\frac{2}{3}}\alpha)^2} = \frac{4(1+\tan^2\alpha)}{(1+\tan^{\frac{2}{3}}\alpha)^3}$$

$$= \frac{4}{(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^3}; \quad \therefore\ m = \frac{2}{(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^{\frac{3}{2}}}.$$

14. If $\alpha + \beta + \gamma = \pi$, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1.$$

The equation may be written

$\cos\alpha(\cos\alpha + \cos\beta\cos\gamma) + \cos\beta(\cos\beta + \cos\alpha\cos\gamma) = 1 - \cos^2\gamma$,

but $\alpha = \pi - (\beta+\gamma)$; $\therefore \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma$, &c.;

$$\therefore\ \cos\alpha \,.\, \sin\beta\sin\gamma + \cos\beta \,.\, \sin\alpha\sin\gamma = \sin^2\gamma,$$

$$\text{or } \cos\alpha\sin\beta + \sin\alpha\cos\beta = \sin(\alpha+\beta) = \sin\gamma,$$

which is true.

15. If $\alpha + \beta + \gamma = \pi$, then squaring $\sin\frac{1}{2}(\beta+\gamma) = \cos\frac{1}{2}\alpha$,

$$\sin^2\tfrac{1}{2}\alpha + \sin^2\tfrac{1}{2}\beta + \sin^2\tfrac{1}{2}\gamma + 2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma = 1.$$

16. $\dfrac{\cos\alpha + \sin\gamma - \sin\beta}{\cos\beta + \sin\gamma - \sin\alpha} = \dfrac{1+\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\beta}$, if $\alpha+\beta+\gamma = \frac{1}{2}\pi$.

The first member

$$= \frac{\sin(\beta+\gamma) + \sin\gamma - \sin\beta}{\sin(\alpha+\gamma) + \sin\gamma - \sin\alpha} = \frac{4\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}\gamma\cos\frac{1}{2}\beta}{4\cos\frac{1}{2}(\alpha+\gamma)\sin\frac{1}{2}\gamma\cos\frac{1}{2}\alpha}$$

$$= \frac{\cos(\frac{1}{4}\pi - \frac{1}{2}\alpha)}{\cos\frac{1}{2}\alpha} \div \frac{\cos(\frac{1}{4}\pi - \frac{1}{2}\beta)}{\cos\frac{1}{2}\beta} = \frac{1+\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\beta}.$$

17. If $\sin^3 x = \sin(\alpha - x)\sin(\beta - x)\sin(\gamma - x)$, where $\alpha + \beta + \gamma = \pi$,

$$\text{then } \cot x = \cot\alpha + \cot\beta + \cot\gamma$$

$$\operatorname{cosec}^2 x = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma.$$

For $\operatorname{cosec}\alpha \operatorname{cosec}\beta \operatorname{cosec}\gamma = \dfrac{\sin(\alpha - x)}{\sin\alpha\sin x}\cdot\dfrac{\sin(\beta - x)}{\sin\beta\sin x}\cdot\dfrac{\sin(\gamma - x)}{\sin\gamma\sin x}$

$$= (\cot x - \cot\alpha)(\cot x - \cot\beta)(\cot x - \cot\gamma);$$

$$\therefore\ \cot^3 x - (\cot\alpha + \cot\beta + \cot\gamma)\cot^2 x$$
$$+ (\cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma)\cot x$$
$$- (\cot\alpha\cot\beta\cot\gamma + \operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma) = 0,$$

or (see Art. 56)

$$\cot^3 x - (\cot\alpha + \cot\beta + \cot\gamma)(1 + \cot^2 x) + \cot x = 0;$$

$\therefore$ rejecting the factor $1 + \cot^2 x$, $\cot x = \cot\alpha + \cot\beta + \cot\gamma$;

$$\therefore\ \cot^2 x = \cot^2\alpha + \cot^2\beta + \cot^2\gamma + 2;$$

$$\therefore\ \operatorname{cosec}^2 x = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma.$$

18. Find θ from the equation $\sin^2 2\theta - \sin^2\theta = \frac{1}{4}$.

Let $\sin\theta = x$; $\therefore\ (2x\sqrt{1 - x^2})^2 - x^2 = \frac{1}{4}$;

$$\therefore\ 16x^4 - 12x^2 + 1 = (4x^2 - 1)^2 - 4x^2 = 0,$$

$$\text{or } (4x^2 - 1 + 2x)(4x^2 - 1 - 2x) = 0;$$

$$\therefore\ x = \tfrac{1}{4}(1 \pm \sqrt{5}), \quad \text{or } \tfrac{1}{4}(-1 \pm \sqrt{5}),$$

$$\text{or } x = \pm\tfrac{1}{4}(\sqrt{5} - 1), \quad \text{or } \pm\tfrac{1}{4}(\sqrt{5} + 1);$$

$$\therefore\ \sin\theta = \sin\left(\pm\frac{\pi}{10}\right), \quad \text{or } \sin\left(\pm\frac{3\pi}{10}\right);$$

$$\therefore\ \theta = n\pi \pm \frac{\quad}{10} \quad \text{or } n\pi \pm \frac{3\pi}{10},$$

both of which are included in $\theta = \left(n \pm \dfrac{1}{5} \pm \dfrac{1}{10}\right)\pi$,

n being any positive or negative integer, and the signs being combined in any manner. Similarly in the equation

$$\tan\theta + \cot\theta = 4, \quad \theta = \tfrac{1}{2}n\pi + (-1)^n\cdot\frac{1}{12}\pi,$$

which may be otherwise expressed $\left(n + \dfrac{1}{4} \pm \dfrac{1}{6}\right)\pi$.

19. Find θ from the equation $\tan^3\theta = \tan(\theta - \alpha)$, and shew that it will not be real unless $\sin\alpha$ be not greater than $\frac{1}{3}$.

$$\tan^2\theta = \frac{\tan(\theta-\alpha)}{\tan\theta}; \quad \therefore \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{\tan\theta - \tan(\theta-\alpha)}{\tan\theta + \tan(\theta-\alpha)},$$

$$\text{or } \cos 2\theta = \frac{\sin\alpha}{\sin(2\theta-\alpha)}, \text{ or } 2\sin\alpha = 2\sin(2\theta-\alpha)\cos 2\theta$$

$$= \sin(4\theta-\alpha) - \sin\alpha;$$

$$\therefore \sin(4\theta-\alpha) = 3\sin\alpha, \text{ and } \theta = \tfrac{1}{4}\{\alpha + \sin^{-1}(3\sin\alpha)\}.$$

20. Find θ from the equation

$$\sin\theta \,.\, \sin(2\alpha+\theta) + n\cos^2\alpha = 0,$$

and shew that it cannot be real unless $\cos\alpha < \dfrac{1}{\sqrt{1+n}}$.

21. Find x in an algebraical form from the equation

$$\sec^{-1}a + \sec^{-1}\frac{x}{a} = \sec^{-1}b + \sec^{-1}\frac{x}{b}. \qquad x^2 = a^2b^2.$$

Find an algebraical relation between x and y from the equation

$$\sin^{-1}\frac{x}{a} - \sin^{-1}\frac{y+b}{\sqrt{a^2+b^2}} + \sin^{-1}\frac{b}{\sqrt{a^2+b^2}} = \frac{\pi}{2}.$$

$$\sin^{-1}\frac{y+b}{\sqrt{a^2+b^2}} = \sin^{-1}\frac{b}{\sqrt{a^2+b^2}} - \cos^{-1}\frac{x}{a}$$

$$= \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\cdot\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{a}\cdot\frac{a}{\sqrt{a^2+b^2}}\right);$$

$$\therefore y + b = \frac{bx}{a} - \sqrt{a^2-x^2}.$$

22. Find θ from the equation

$$\tan\theta + 2\cot 2\theta = \sin\theta\,(1 + \tan\theta\tan\tfrac{1}{2}\theta),$$

$$\text{or } \tan\theta + 2\,.\,\frac{1-\tan^2\theta}{2\tan\theta} = \cot\theta = \sin\theta\,(1+\tan\theta\tan\tfrac{1}{2}\theta);$$

$$\therefore \cot^2\theta = \sin\theta\,(\cot\theta + \tan\tfrac{1}{2}\theta) = \cos\theta + 2\sin^2\tfrac{1}{2}\theta = 1;$$

$$\therefore \theta = n\pi \pm \frac{\pi}{4}.$$

23. To shew *a priori* that $\tan a$, expressed in terms of $\sin 4a$, will have four values, and to find them.

Suppose that $\tan a = F(\sin 4a)$, then because in the second member we may write $n\pi + (-1)^n . 4a$ for $4a$ without altering its value, the first member must be the value of

$$\tan\{\tfrac{1}{4}n\pi + (-1)^n . a\},$$

which expression (n being any integer, and therefore of one of the forms $4m$, $4m+1$, $4m+2$, $4m+3$) can only have the four values $\tan a$, $\tan(\frac{1}{4}\pi - a)$, $\tan(\frac{1}{2}\pi + a)$, $\tan(\frac{3}{4}\pi - a)$.

To find these values in terms of $\sin 4a$, let $x = \tan a$, and

$$\frac{1}{a} = \sin 4a = 2\sin 2a\cos 2a$$

$$= 2\frac{2\tan a}{1+\tan^2 a} . \frac{1-\tan^2 a}{1+\tan^2 a} = \frac{4x(1-x^2)}{(1+x^2)^2};$$

$$\therefore\ (x^2+1)^2 + 4ax(x^2-1) = 0,$$

$$\text{or}\ \left(x - \frac{1}{x}\right)^2 + 4a\left(x - \frac{1}{x}\right) + 4a^2 = 4(a^2-1);$$

$$\therefore\ x - \frac{1}{x} + 2a = \pm 2\sqrt{a^2-1};$$

$$\therefore\ \tan a = x = \{\sqrt{a+1} - \sqrt{a}\} . \{-\sqrt{a-1} + \sqrt{a}\}$$

$$= \frac{(\sqrt{1+\sin 4a} - 1)(-\sqrt{1-\sin 4a} + 1)}{\sin 4a},$$

which expression contains implicitly the four values.

24. If $\cos\theta = \cos^2 a - \sin^2 a\sqrt{1 - c^2\sin^2\theta}$, then θ is given by the equation $\tan\frac{1}{2}\theta = \tan a\sqrt{1 - c^2\sin^2 a}$.

$$1 - \cos\theta = \sin^2 a(1 + \sqrt{1 - c^2\sin^2\theta}) = \frac{c^2\sin^2\theta\sin^2 a}{1 - \sqrt{1 - c^2\sin^2\theta}};$$

$$\therefore\ c^2\sin^4 a(1+\cos\theta) = \sin^2 a(1 - \sqrt{1 - c^2\sin^2\theta});$$

$$\therefore\ 1 + c^2\sin^4 a - \cos\theta(1 - c^2\sin^4 a) = 2\sin^2 a;$$

$$\therefore\ \cos\theta = \frac{1 - 2\sin^2 a + c^2\sin^4 a}{1 - c^2\sin^4 a}$$

$$\therefore\ \tan^2\tfrac{1}{2}\theta = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2 a(1 - c^2\sin^2 a).$$

25. If $\sec\alpha\sec\theta + \tan\alpha\tan\theta = \sec\beta$,

then $\sec\alpha\tan\theta + \tan\alpha\sec\theta = \tan\beta$.

Squaring the first,

$$\sec^2\alpha\sec^2\theta + 2\sec\alpha\tan\alpha\sec\theta\tan\theta + \tan^2\alpha\tan^2\theta = \sec^2\beta,$$

$$\text{but } \sec^2\alpha \qquad - \tan^2\alpha \quad = \quad 1;$$

$$\therefore\ \sec^2\alpha\tan^2\theta + 2\sec\alpha\tan\alpha\sec\theta\tan\theta + \tan^2\alpha\sec^2\theta = \tan^2\beta,$$

$$\text{or } \sec\alpha\tan\theta + \tan\alpha\sec\theta = \tan\beta.$$

26. $\tan 9^0 = \sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}}.$

Since $\cos 18^0 = \sqrt{1 - \frac{1}{16}(\sqrt{5} - 1)^2} = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$,

$$\tan 9^0 = \frac{1 - \cos 18^0}{\sin 18^0}$$

$$= \frac{4 - \sqrt{10 + 2\sqrt{5}}}{\sqrt{5} - 1} = \sqrt{5} + 1 - \sqrt{5 + 2\sqrt{5}}.$$

27. Given the areas A and B of the regular polygons inscribed in and circumscribed about a circle, to find the areas a and b of the regular polygons of double the number of sides, inscribed in and circumscribed about the same circle.

If r be the radius of the circle

$$A = \tfrac{1}{2}nr^2\sin\frac{2\pi}{n} = nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}, \quad B = nr^2\tan\frac{\pi}{n};$$

$$\therefore\ a = \tfrac{1}{2}2nr^2\sin\frac{2\pi}{2n} = nr^2\sin\frac{\pi}{n} = \sqrt{AB},$$

$$b = 2nr^2\tan\frac{\pi}{2n} = \frac{2nr^2\sin\frac{\pi}{n}}{1 + \cos\frac{\pi}{n}} = \frac{2\sqrt{AB}}{1 + \sqrt{\frac{A}{B}}} = \frac{2B\sqrt{A}}{\sqrt{A} + \sqrt{B}}.$$

28. Shew that

$$\tan^{-1}\frac{t_1 - t_2}{1 + t_1t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2t_3} + \&c. = \tan^{-1}t_1 - \tan^{-1}t_n.$$

$$\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2},$$

$$\tan^{-1} t_2 - \tan^{-1} t_3 = \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3},$$

....

$$\tan^{-1} t_{n-1} - \tan^{-1} t_n = \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n};$$

$$\therefore \tan^{-1} t_1 - \tan^{-1} t_n = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} + \&c.$$

$$+ \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

29. Prove that $\tan 2a = 2 \tan \{a + \tan^{-1} (\tan^3 a)\}$.

$$\text{The 2nd member} = 2 . \frac{\tan a + \tan^3 a}{1 - \tan^4 a} = \frac{2 \tan a}{1 - \tan^2 a}.$$

30. To determine an angle (θ) less than a right angle whose circular measure equals its cotangent.

Since $\frac{1}{4}\pi = {\cdot}7854$, and $\cot \frac{1}{4}\pi = 1$, we shall have $\theta = \frac{1}{4}\pi + a$, where a is a small fraction; therefore if $\frac{1}{4}\pi = 1 - x$,

$$1 - x + a = \cot (\tfrac{1}{4}\pi + a) = \frac{1 - \tan a}{1 + \tan a},$$

or $a - x + (2 + a - x) \tan a = 0$; or, putting $\tan a = a + \frac{1}{3} a^3$,

$$3a + a^2 + \tfrac{2}{3} a^3 = x (1 + a + \tfrac{1}{3} a^3).$$

Let $a = \frac{1}{3} x + b x^2 + c x^3$; $\therefore x + \left(3b + \frac{1}{9}\right) x^2 + \&c. = x + \frac{1}{3} x^2 + \&c.$;

$$\therefore 3b + \frac{1}{9} = \frac{1}{3}, \quad \text{or } b = \frac{2}{27}, \quad \text{and } c = 0;$$

$$\therefore \theta = \frac{1}{4}\pi + a = \frac{1}{4}\pi + \frac{1}{3}\left(1 - \frac{1}{4}\pi\right) + \frac{2}{27}\left(1 - \frac{1}{4}\pi\right)^2$$

$$= {\cdot}7854 + {\cdot}07153 + {\cdot}00341 = {\cdot}86034;$$

$$\therefore A = \theta \times 57^\circ . 29577 = 49^\circ . 17' \text{ nearly}.$$

31. To approximate to the positive values of θ in the equation $\tan\theta = \theta$.

Every value of θ will evidently be of the form $n\,.\,\frac{\pi}{2} - a$, where n is an odd number greater than 1, and a is a small fraction that decreases rapidly as n increases. Hence

$$\tfrac{1}{2}\,n\,\pi - a = \cot a, \quad \text{or, putting } x = \frac{2}{n\pi},$$

$$\tan a\,(1 - a x) - x = 0; \quad \text{or, since } a \text{ is small,}$$

$$\left(a + \frac{1}{3}a^3 + \frac{2a^5}{3\,.\,5}\right)(1 - a x) - x = 0,$$

$$\text{or } a + \frac{1}{3}a^3 + \frac{2a^5}{15} = x\,(1 + a^2 + \frac{1}{3}a^4).$$

Assume $a = x + a x^3 + b x^5$;

$$\therefore\ x + \left(a + \frac{1}{3}\right)x^3 + \left(b + a + \frac{2}{15}\right)x^5 + \&c.$$

$$= x + x^3 + \left(2a + \frac{1}{3}\right)x^5 + \&c.;$$

$$\therefore\ a = \frac{2}{3} \text{ and } b = \frac{13}{15}; \text{ and } a = x\left(1 + \frac{2x^2}{3} + \frac{13}{15}x^4\right),$$

$$\text{and } \theta = \tfrac{1}{2}\,n\,\pi - \frac{2}{n\pi}\left\{1 + \frac{2}{3}\left(\frac{2}{n\pi}\right)^2 + \frac{13}{15}\left(\frac{2}{n\pi}\right)^4 + \&c.\right\},$$

n being any odd number > 1. If $n = 3$, then $\theta = 4{\cdot}4934118$.

32. To determine the angle θ whose circular measure equals its cosine.

Since $\frac{1}{4}\pi = {\cdot}7854$, and $\cos\frac{1}{4}\pi = {\cdot}7071 = c$ suppose, therefore $\theta = \frac{1}{4}\pi - a$, where a is a small fraction. Then if $\frac{1}{4}\pi = c + x$, where $x = {\cdot}0783$, we have

$$c + x - a = \cos(\tfrac{1}{4}\pi - a) = c(\cos a + \sin a) = c(1 + a - \tfrac{1}{2}a^2) \text{ nearly},$$

$$\therefore\ x = a\,(1 + c - \tfrac{1}{2}\,c a);$$

therefore, for a first approximation, $a = \dfrac{x}{1+c}$; and more nearly,

$$a = \frac{x}{1+c}\left\{1 + \frac{c a}{2\,(1+c)}\right\} = \frac{x}{1+c}\left\{1 + \frac{c x}{2\,(1+c)^2}\right\}.$$

$$\therefore\ \theta = \tfrac{1}{4}\pi - \frac{x}{1+c} - \frac{1}{2}\frac{cx^2}{(1+c)^3} = \cdot 7854 - \cdot 0458 - \cdot 00044$$

$$= \cdot 7391\,;$$

$\therefore$ θ corresponds to $\cdot 7391 \times 57^0 . 29577 = 42^0 . 20' . 47'',259$.

33. To prove that

$$2 \sin 30^0 \left\{1 - (-\tfrac{1}{2})^n\right\} = \sqrt{2 - \sqrt{2 - \&c.}} \text{ to } n \text{ terms.}$$

For A write $45^0 - x$ in the formula $2 \sin A = \sqrt{2 - 2 \cos 2A}$;

$$\therefore\ 2 \sin (45^0 - x) = \sqrt{2 - 2 \sin 2x}\,;$$

put $2x = 45^0$, $(1 - \frac{1}{2})\,45^0$, $(1 - \frac{1}{2} + \frac{1}{4})\,45^0$, &c. successively;

$$\therefore\ 2 \sin (1 - \tfrac{1}{2})\,45^0 = \sqrt{2 - \sqrt{2}},$$

$$2 \sin (1 - \tfrac{1}{2} + \tfrac{1}{4})\,45^0 = \sqrt{2 - \sqrt{2 - \sqrt{2}}}, \text{ \&c.}$$

$$2 \sin \left\{1 - \tfrac{1}{2} + \tfrac{1}{4} - \&c. + (-\tfrac{1}{2})^{n-1}\right\} 45^0 = \sqrt{2 - \sqrt{2 - \&c.}}\,;$$

$$\text{or } 2 \sin \tfrac{2}{3} \left\{1 - (-\tfrac{1}{2})^n\right\} 45^0 = \sqrt{2 - \sqrt{2 - \&c.}} \text{ to } n \text{ terms.}$$

$$\text{Similarly, } 2 \cos \frac{45^0}{2^n} = \sqrt{2 + \sqrt{2 + \&c.}} \text{ to } n \text{ terms.}$$

34. Find the angles of a triangle where $3c = 7b$, and $\angle A = 6^0 . 37' . 24''$; having given

$$\log 2 = \cdot 3010300,\quad L \tan 3^0 . 18' . 42'' = 8 \cdot 7624069,$$

$$L \tan 8^0 . 13' . 50'' = 9 \cdot 1603083, \text{ diff. for } 10'' = 1486.$$

$$\tan (\tfrac{1}{2}A + B) = \frac{c+b}{c-b} \tan \frac{A}{2} = \frac{10}{4} \tan \tfrac{1}{2}A\,;$$

$$\therefore\ L \tan (\tfrac{1}{2}A + B) = 1 - 2 \log 2 + L \tan \tfrac{1}{2}A$$

$$= 1 - \cdot 6020600 + 8 \cdot 7624069 = 9 \cdot 1603469\,;$$

$$\therefore\ \tfrac{1}{2}A + B = 8^0 . 13' . 52'',6\,; \qquad \therefore\ B = 4^0 . 55' . 10'',6.$$

35. The angle which the Earth's radius subtends at the Sun being $8'',57116$, find the distance of the Sun from the Earth expressed in terms of the Earth's radius.

Let r be the radius of a section of the Earth, made by a plane through its center perpendicular to the line joining its center with the Sun's center; then if θ be the circular measure

of the angle which r subtends at the Sun's center, and d = distance of centers,

$$\frac{r}{d} = \tan\theta = \theta \text{ nearly, since } \theta \text{ is very small,}$$

$$\frac{8'',57116}{57^0.29577} \quad \frac{0\cdot00238}{57\cdot29577}\cdot$$

$\therefore\ d = r \times 24074$ = about 24 thousand times the Earth's radius.

Similarly, the angle which r subtends at the Moon being $57'.1'',8$, the Moon's distance = 60,2796 times of the Earth's radius.

Having given that two points, each 10 feet above the surface, cease to be visible from each other over still water at a distance of 8 miles, find the Earth's diameter.

Let A, B, (fig. 36) be the two points, then AB is a tangent to the Earth's surface at its middle point D; $AD = DE$ nearly = 4 miles, and AE = 10 feet = $\dfrac{10}{3 \times 1760}$ miles. Let C be the Earth's center, and $CD = r$, then $AE\,.\,(2r + AE) = AD^2$, or $2r\,.\,\dfrac{10}{3 \times 1760}$ 16 nearly; $\therefore\ r = 4224$ miles.

36. Find the number of minutes and seconds in the angle which a flag-staff 5 yards long and standing on the top of a tower two hundred yards high, subtends at a point in the horizontal plane 100 yards from the foot of the tower.

Draw PC (fig. 37) perpendicular to AC, and let θ be the circular measure of the angle PAC; then since θ is small,

$$\theta = \tan\theta = \frac{PC}{AC} = \frac{CD\cos BAC}{AB\sec BAC} \quad 20\,(1 + \tan^2 BAC) \quad \frac{1}{100};$$

$$\angle PAC = \frac{57^0.29577}{100} \quad 0^0.5729577 = 34'.22'',64.$$

37. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 to 5. Shew that its largest angle equals 120^0.

If $6a$ be the perimeter of the equilateral triangle, its area

$= \frac{1}{2} 4a^2 \sin 60^0 = a^2\sqrt{3}$; let $2a - x$, $2a$, $2a + x$, be the sides of the other triangle, then its

$$(\text{area})^2 = 3a\,(a + x)\,a\,(a - x) = \left(\frac{3a^2\sqrt{3}}{5}\right)^2;$$

$$\therefore\ a^2 - x^2 = \frac{9}{25}a^2, \quad \text{or } x = \frac{4a}{5}.$$

Let A be the greatest angle which is opposite $2a + x$,

$$\text{then } \sin A = \frac{2\text{ area}}{2a\,(2a - x)} = \frac{\frac{3}{5}a^2\sqrt{3}}{a\,\frac{6a}{5}} = \tfrac{1}{2}\sqrt{3}.$$

38. Having given the perimeter $(2s)$ and the three angles of a triangle, to find the sides.

$$\frac{s - c}{s} = \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B;\ \therefore\ c = s\,(1 - \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B).$$

39. The angles of elevation α, β, γ, of an object above a horizontal plane are taken at three known stations in the same straight line in that plane; to determine its altitude.

Let the altitude $OD = x$ (fig. 38), then $AD = x\cot\alpha$, $DB = x\cot\beta$, $DC = x\cot\gamma$; also let $AB = a$, $BC = b$; then

$$\cos DBC = \frac{b^2 + x^2\cot^2\beta - x^2\cot^2\gamma}{2bx\cot\beta} = -\frac{a^2 + x^2\cot^2\beta - x^2\cot^2\alpha}{2ax\cot\beta},$$

$$\therefore\ x^2 = \frac{ab\,(a + b)}{a\cot^2\gamma + b\cot^2\alpha - (a + b)\cot^2\beta}.$$

40. A circle may be described through the centers of four circles, each of which touches one side and the two adjacent ones produced of any quadrilateral. Let the lines bisecting the exterior angles meet in O, O_1, O_2, O_3, (fig. 39), these are the centers of the four circles.

Now $\angle O = 180^0 - \frac{1}{2}(A + D)$, ($A$ and D denoting the exterior angles),

$$\angle O_2 = 180^0 - \tfrac{1}{2}(B + C);$$

$$\therefore\ O + O_2 = 360^0 - \tfrac{1}{2}(A+B+C+D) = 180^0,$$

since all the exterior angles of any rectilinear figure are together equal to four right angles. Hence a circle may be described through O, O_1, O_2, O_3.

41. When a quadrilateral is capable of having both a circle inscribed in it and one circumscribed about it, its area equals the square root of the product of its sides.

Suppose the inscribed circle to touch the sides in the points a, b, c, d (fig. 27, Plate I.), then

$$Ad = Aa,\quad Dd = Dc,\quad Cb = Cc,\quad Bb = Ba;$$

$\therefore$ adding these equations together, we get

$$AD + BC = AB + DC, \text{ or } a + c = b + d;$$

$$\therefore \tfrac{1}{2} \text{ perimeter } s = a + c, \text{ or } = b + d;$$

$$\therefore \text{ area } = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{abcd}.$$

42. To construct a triangle similar to a given triangle, whose angular points shall be in three given parallel straight lines; and to find expressions for its sides.

Take any point D in LK (fig. 40), make angles LDC, KDB, equal to two of the given angles; therefore BDC is equal to the third.

Join BC, and about $\triangle DBC$ describe a circle cutting LK in A; then ABC is the required triangle; for $\angle ABC = ADC$, and $\angle CAB = CDB$.

Also $\quad BC^2 = BD^2 + DC^2 - 2BD \cdot DC \cos A$

$$= \left(\frac{c}{\sin C}\right)^2 + \left(\frac{b}{\sin B}\right)^2 - \frac{2bc}{\sin B \sin C}\cos A,$$

since $DC \sin B = LH = b$, $BD \sin C = ML = c$.

43. If an equilateral triangle have its angular points in three parallel straight lines of which the middle one is distant from the outside ones by a, c, its side will $= 2\sqrt{\dfrac{a^2 + c^2 + ac}{3}}$.

44. In any triangle, the point of intersection of the perpendiculars from the angles on the opposite sides, the point of intersection of the lines perpendicular to the sides through their middle points, and the point of intersection of the lines

drawn bisecting the sides from the opposite angles, are in a straight line.

P the intersection of the perpendiculars (fig. 41),

O the center of the circumscribed circle,

OM perpendicular to CB. Join AM, OP, cutting one another in G;

$$\text{then } OM = OC \cos A = \frac{c}{2 \sin C} \cos A = \frac{AQ}{2 \sin C} = \tfrac{1}{2} AP;$$

$$\therefore MG = \tfrac{1}{2} AG;$$

$\therefore$ G is the point of intersection of the lines drawn bisecting the sides from the opposite angles.

45. If from the angles A, B, C of a triangle three lines be drawn through any point O to meet the opposite sides in α, β, γ, then the continued product of the alternate segments of the sides are equal to one another, that is, (fig. 42),

$$A\beta \, . \, C\alpha \, . \, B\gamma = A\gamma \, . \, B\alpha \, . \, C\beta.$$

Since triangles on the same base are to one another as their altitudes,

$$\frac{\triangle AOB}{\triangle COB} = \frac{A\beta}{C\beta}, \quad \frac{\triangle AOC}{\triangle AOB} = \frac{C\alpha}{B\alpha}, \quad \frac{\triangle BOC}{\triangle AOC} = \frac{B\gamma}{A\gamma};$$

$$\therefore \frac{A\beta}{C\beta} \cdot \frac{C\alpha}{B\alpha} \cdot \frac{B\gamma}{A\gamma} = 1, \text{ or } A\beta \, . \, C\alpha \, . \, B\gamma = C\beta \, . \, B\alpha \, . \, A\gamma.$$

Hence if three points in the sides of a triangle can be shewn to satisfy this condition, the lines joining them with the opposite angles will intersect in the same point. The condition is satisfied in the following cases.

(1) Let α, β, γ be the feet of the perpendiculars dropped from the angles upon the opposite sides, then each side of the above equation equals $abc \cos A \cos B \cos C$.

(2) Let α, β, γ be the intersections of the lines bisecting the angles with the opposite sides, then $A\gamma = \dfrac{bc}{a + b}$, and each side of the equation equals $\dfrac{a^2 b^2 c^2}{(a + b)(a + c)(b + c)}$.

(3) Let α, β, γ be the middle points of the sides; then the condition is evidently satisfied.

(4) Let a, β, γ be the points of contact of the inscribed circle, then $A\beta = A\gamma$, and the condition is evidently satisfied

(5) Let a, β, γ be the points of contact of the circles touching one side, and the two others produced; then $A\gamma = s - b$, s being the semi-perimeter; therefore $A\gamma \,.\, C\beta \,.\, Ba = (s-b)$ $(s-a)$ $(s-c)$, and the condition is satisfied.

The above condition may be also expressed as follows:

$$\frac{A\beta}{C\beta} = \frac{AB \sin ABO}{BC \sin CBO}, \quad \frac{Ca}{Ba} = \frac{AC \sin CAO}{AB \sin BAO}, \quad \frac{B\gamma}{A\gamma} = \frac{BC \sin BCO}{AC \sin ACO},$$

$$\therefore \ 1 = \frac{\sin ABO \sin CAO \sin BCO}{\sin CBO \sin BAO \sin ACO}.$$

46. If from the angles of a triangle three lines Aa, Bb, Cc, (fig. 43) be drawn to meet the opposite sides, and from any point O within the triangle three lines Oa, $O\beta$, $O\gamma$ be drawn parallel to them to meet the sides,

$$\text{then } \frac{Oa}{Aa} + \frac{O\beta}{Bb} + \frac{O\gamma}{Cc} = 1.$$

For if p, p', q, q', r, r', be perpendiculars from A and O upon BC, from B and O upon AC, and from C and O upon AB; and S = area of the triangle, then

$$S = \triangle BOC + \triangle AOC + \triangle AOB \quad \frac{S}{p} \,.\, p' + \frac{S}{q} \,.\, q' + \frac{S}{r} \,.\, r';$$

$$\text{but } \frac{p'}{p} = \frac{Oa}{Aa}, \text{ \&c.}; \qquad \frac{Oa}{Aa} + \frac{O\beta}{Bb} + \frac{O\gamma}{Cc} = 1.$$

If O be the intersection of the lines Aa, Bb, Cc, the equation becomes

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

47. Having given the lengths a, b, c of the chords of three arcs BC, CD, DA, which together make up a semicircle, to find the diameter $AB = x$. We have square of diagonal BD

$$= a^2 + b^2 - 2ab \cos C = a^2 + b^2 + 2ab \cos A = a^2 + b^2 + 2ab \,.\, \frac{c}{x},$$

also $= x^2 - c^2$; $\therefore$ $x^3 - (a^2 + b^2 + c^2)\, x - 2abc = 0$, the equation for finding x.

48. Every two angles of a pentagon are joined, and the areas of the five triangles so formed, each of which has two sides in common with the pentagon, are given to find the area of the pentagon.

Let $AB = t$, $AC = u$, $AD = v$, $AE = w$ (fig. 44);

$$\angle BAC = \theta,\ BAD = \phi,\ \angle BAE = \psi,$$

$$2\triangle ABC = tu \sin\theta,\ 2\triangle ABD = tv\sin\phi,\ 2\triangle AEB = tw\sin\psi,$$

$$2\triangle AED = vw\sin(\psi - \phi),\ 2\triangle ACE = uw\sin(\psi - \theta),$$

$$2\triangle ACD = uv\sin(\phi - \theta);$$

$$\therefore\ 4\triangle ABC \times \triangle AED + 4\triangle AEB \times \triangle ACD$$

$$= tuvw\left\{\sin\theta\sin(\psi - \phi) + \sin\psi\sin(\phi - \theta)\right\}$$

$$= tuvw\sin\phi\sin(\psi - \theta) = 4\triangle ABD \times \triangle ACE.$$

Now call the areas of the triangles ABC, BCD, CDE, DEA, EAB, a, b, c, d, e; and the areas of the triangles ABD, ACD, ACE, x, y, z; and the area of the pentagon p; then the result just obtained is $ad + ey = xz$, and we have

$$b + d + x = p,\quad a + d + y = p,\quad a + c + z = p;$$

hence eliminating x, y, z, we get, to determine p, the equation

$$p^2 - (a + b + c + d + e)\,p + (ab + bc + cd + de + ea) = 0.$$

49. In the sides of a given triangle ABC take two points M and N, and in the line joining them take a point P, (fig. 45) such that

$$\frac{MB}{AM} = \frac{AN}{NC} = \frac{PM}{PN};$$

prove that if CP, BP, be joined, triangle $BPC = 2$ triangle MAN.

$$\frac{\triangle ABC}{\triangle AMN} = \frac{AB}{AM}\cdot\frac{AC}{AN} = \frac{(n+1)^2}{n},\ \text{if}\ \frac{BM}{AM}$$

$$\frac{\triangle BMP}{\triangle AMN} = \frac{BM}{AM}\cdot\frac{MP}{MN} = \frac{n^2}{n+1},$$

$$\frac{\triangle PNC}{\triangle AMN} = \frac{NC}{AN}\cdot\frac{PN}{MN} = \frac{1}{n(n+1)};$$

hence subtracting the sum of the two latter equations from the former

$$1 + \frac{\Delta BPC}{\Delta MAN} = \frac{(n+1)^3 - (n^3+1)}{n(n+1)} = 3, \text{ or } \Delta BPC = 2\Delta MAN.$$

Taking the product of these three equations we get

$$\Delta ABC \,.\, \Delta BMP \,.\, \Delta PNC = (\Delta AMN)^3.$$

Also

$$\sqrt[3]{\Delta BPM} + \sqrt[3]{\Delta PNC} = \frac{(n-1)^{\frac{2}{3}}}{\sqrt[3]{n}} \,.\, \sqrt[3]{\Delta AMN} = \sqrt[3]{\Delta ABC}.$$

50. The area of any triangle is to the area of the triangle whose sides are respectively equal to the lines joining its angular points with the middle points of the opposite sides as 4 to 3.

Through D and E (fig. 46), the middle points of AB, BC, draw DF, GEF, respectively parallel to AE, AB, and join CF, BF, DE; then because AF is a parallelogram, $DF = AE$, and EF is parallel and equal to AD or DB, and therefore BF is parallel and equal to DE or HC, and consequently FC is equal to HB. Hence DCF is the required triangle, and its area equals

$$\Delta DCB \times \frac{GF}{DB} = \Delta DCB \times \left(1 + \frac{GE}{DB}\right) = \frac{3}{2}\Delta DCB = \frac{3}{4}\Delta ACB.$$

51. The area of any triangle is to the area of the triangle formed by joining the points where the lines bisecting the angles meet the opposite sides, as $(a+b)(a+c)(b+c)$ to $2abc$.

Since $\dfrac{A\gamma}{B\gamma} = \dfrac{b}{a}$ (fig. 42), $A\gamma = \dfrac{bc}{a+b}$, and similarly $A\beta = \dfrac{bc}{a+c}$;

$$\therefore\ \Delta A\beta\gamma = \frac{1}{2}\frac{b^2c^2}{(a+b)(a+c)}\sin A = \Delta ABC \,.\, \frac{bc}{(a+b)(a+c)};$$

$$\therefore\ \Delta ABC = \Delta\alpha\beta\gamma + \Delta ABC\left\{\frac{bc}{(a+b)(a+c)} + \frac{ac}{(a+b)(b+c)} + \frac{ab}{(a+c)(b+c)}\right\};$$

$$\therefore\ \Delta\alpha\beta\gamma = \Delta ABC \,.\, \frac{2abc}{(a+b)(a+c)(b+c)}.$$

52. The area of any acute-angled triangle ABC (fig. 47) is to the area of the triangle $A'B'C'$ formed by joining the feet of the perpendiculars dropped from the angles upon the opposite sides as 1 to $2 \cos A \cos B \cos C$.

Since

$$\Delta AB'C' = \tfrac{1}{2} AC' . AB' \sin A = \tfrac{1}{2} bc \sin A \cos^2 A = \Delta ABC \cos^2 A;$$

$$\therefore \ \Delta ABC = \Delta A'B'C' + \Delta ABC (\cos^2 A + \cos^2 B + \cos^2 C)$$

$$= \Delta A'B'C' + \Delta ABC (1 - 2 \cos A \cos B \cos C); \text{ (Prob. 14)}$$

$$\therefore \ \Delta A'B'C' = \Delta ABC \times 2 \cos A \cos B \cos C.$$

If the triangle have an obtuse angle B, then

$$\Delta ABC = - \Delta A'B'C' + \Delta ABC (\cos^2 A + \cos^2 B + \cos^2 C);$$

and the ratio of the triangles is $1 : -2 \cos A \cos B \cos C$.

53. If perpendiculars be dropped from the angles upon the opposite sides of an acute-angled triangle and the feet of the perpendiculars be joined, a triangle will be formed having the angular points of the original triangle and the intersection of the perpendiculars for the centers of four circles which touch its sides.

We have $\angle ABB' = ACC'$ being complements of the same angle BAC; but $ABB' = OA'C'$, because they would be in the same segment of the circle described about $OA'BC'$; similarly $ACC' = OA'B'$; $\therefore$ $OA'C' = OA'B'$; hence O is the center of the circle inscribed in $\Delta A'B'C'$; also the exterior angles formed by producing $A'B'$, $A'C'$ are bisected by AB', AC'; therefore A is the center of the circle which touches one side $C'B'$ and the two others produced of the triangle $A'B'C'$.

$$\text{Also } \angle A = B'OC = B'A'C = \tfrac{1}{2}(\pi - A'); \text{ hence (Prob. 52)}$$

$$\Delta A'B'C' = \Delta ABC . 2 \sin \tfrac{1}{2} A' \sin \tfrac{1}{2} B' \sin \tfrac{1}{2} C',$$

which is the relation between the area of a triangle, and the area of the triangle whose angles are in the centers of the circles, each of which touches its three sides.

$$\text{Also } \frac{B'C'}{BC} = \frac{CC'}{BC} . \frac{B'C'}{CC'} = \sin B . \frac{\sin ACC'}{\sin ABC} = \cos A,$$

$$\text{or } B'C' = BC \cos A.$$

Again let R, R', be the radii of the circles circumscribed about the triangles ABC, $A'B'C'$,

$$\text{then } R = \tfrac{1}{2}\frac{BC}{\sin A} = \tfrac{1}{2}\frac{B'C'}{\sin A\cos A} = \frac{B'C'}{\sin A'} = 2R'.$$

Also since $\angle BOC + A = \pi$, the circles described about ΔABC and ΔBOC would be equal; hence if four circles touch each three sides of a given triangle, and a circle pass through the centers of any three of them, its radius is double of the radius of the circle which circumscribes the triangle.

54. The perimeter of the triangle formed by joining the feet of the perpendiculars dropped from the angles upon the opposite sides of a triangle, or those sides produced, is less than the perimeter of any other triangle whose angular points are in the sides of the first, or in those sides produced.

For two lines drawn from two given points to meet in a line given in position, have their sum the least possible, when they make equal angles with the line. If therefore there be a triangle inscribed in ABC such that its sides do not make equal angles with the three sides of ABC, then another triangle may be inscribed in ABC whose perimeter is less; and if its sides do make equal angles with the sides of ABC, then it coincides with $A'B'C'$; therefore $A'B'C'$ has its perimeter the least possible. And the perimeter of $A'B'C'$

$$= a\cos A + b\cos B + c\cos C = \frac{8\,(\text{area})^2}{abc}.$$

55. If R be the radius of a circle circumscribing a triangle, r the radius of the circle inscribed in it, and r' the radius of the circle touching the side c and the two others produced, then

$$r = 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C,\quad r' = 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

$$\text{For } r = \frac{c}{\cot\frac{1}{2}A + \cot\frac{1}{2}B} = \frac{c\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C},$$

$$r' = \frac{c}{\tan\frac{1}{2}A + \tan\frac{1}{2}B} = \frac{c\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C};$$

and if we substitute the value of R $\dfrac{c}{2\sin C}$, we obtain the above results.

56. To find the distance of the center of a circle which touches the three sides of a triangle from the center of the circle which passes through the angular points.

Let Q be the center of the circumscribed circle, and first let the other circle be inscribed, and let P be its center; then (fig. 48)

$$\angle PAN = \tfrac{1}{2} A, \quad \angle QAN = 90^0 - C, \quad \therefore \; PAQ = \tfrac{1}{2}(C - B);$$

$$\therefore \; (PQ)^2 = R^2 + \left(\frac{r}{\sin \frac{1}{2} A}\right)^2 - \frac{2Rr}{\sin \frac{1}{2} A} \cos \tfrac{1}{2}(C - B);$$

$$\text{but } r = 4R \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C;$$

$$\therefore \; (PQ)^2 = R^2 - \frac{2Rr}{\sin \frac{1}{2} A} \left\{\cos \tfrac{1}{2}(C - B) - 2 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C\right\}$$

$$= R^2 - 2Rr.$$

Next let the circle touch one side c, and the two others produced; then (fig. 49)

$$\angle PAN = 90^0 - \tfrac{1}{2} A, \quad QAN = 90^0 - C; \; \therefore PAQ = 90^0 - \tfrac{1}{2}(C - B);$$

$$\therefore \; (PQ)^2 = R^2 + \left(\frac{r'}{\cos \frac{1}{2} A}\right)^2 - \frac{2Rr'}{\cos \frac{1}{2} A} \sin \tfrac{1}{2}(C - B),$$

$$\text{but } r' = 4R \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C;$$

$$\therefore \; (PQ)^2 = R^2 + \frac{2Rr'}{\cos \frac{1}{2} A} \left\{2 \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C - \sin \tfrac{1}{2}(C - B)\right\}$$

$$= R^2 + 2Rr'.$$

57. If $r\ r_1\ r_2\ r_3$ be the radii of the circles which touch the three sides of a triangle, and R the radius of the circle passing through the angular points, then $4R = r_1 + r_2 + r_3 - r$.

Let a, b, c be the sides of the triangle, $2s$ its perimeter and S its area; and S' the area of the triangle whose angles are in the centers of the exterior touching circles; then (Art. 127)

$$r_1(s - b) + r_2(s - c) + r_3(s - a) = 3S = 3rs,$$

$$\text{and } r_1 b + r_2 c + r_3 a + 2rs = 2S' = \frac{4R}{r} S = 4Rs;$$

$$\text{because } S = S' \, 2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C = S' \frac{r}{2R};$$

$$\therefore\ s\,(r_1 + r_2 + r_3) = (4R + r)\,s,$$

$$\text{or }\ 4R = r_1 +_2 + r_3 - r.$$

58. If $d\ d_1\ d_2\ d_3$ be the distances, from the center of the circumscribed circle, of the centers of the circles which touch the three sides of a triangle, and R be the radius of the circumscribed circle, then

$$d^2 + d_1^2 + d_2^2 + d_3^2 = 4R^2 + 2R\,(r_1 + r_2 + r_3 - r) = 12R^2.$$

59. Having given the area of a triangle and the radii of the inscribed and circumscribed circles, to form the equation whose roots shall be the excess of every two sides above the third side.

Let $x^3 - p_1 x^2 + p_2 x - p_3 = 0$ be the equation,

then $p_1 = a + b + c = \dfrac{2S}{r}$

$$p_3 = 4r^3 \cdot \frac{2S}{r} = 8rS$$

$$\frac{2p_2}{p_3} = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{r_1 + r_2 + r_3}{s} = \frac{4R + r}{s},$$

$$\therefore\ p_2 = 4r\,(4R + r).$$

If we put $2y = p_1 - x$, the transformed equation will have a, b, c, for its roots, viz.

$$y^3 - \frac{2S}{r}y^2 + \left(\frac{S^2}{r^2} + 4Rr + r^2\right)y - 4RS = 0.$$

60. The radius of the circumscribed circle of a triangle is greater than the diameter of the inscribed circle.

For $$\left(\frac{R}{2r}\right)^2 = \frac{a^2b^2c^2}{(a+b-c)^2\,(a+c-b)^2\,(b+c-a)^2}$$

$$= \frac{a^2}{a^2 - (b-c)^2} \cdot \frac{b^2}{b^4 - (a-c)^2} \cdot \frac{c^2}{c^2 - (a-b)^2},$$

and each of the latter fractions exceeds unity.

61. The perpendicular distances of the angle A of a triangle from two lines at right angles to one another drawn through

$\angle C$ are $e - \cos\alpha$, $\sqrt{1-e^2}\sin\alpha$; and the perpendicular distances of B from the same lines are $e - \cos\beta$, $\sqrt{1-e^2}\sin\beta$; prove that if $a+b+c = 2(1-\cos\phi)$, $a+b-c = 2(1-\cos\theta)$, then $\alpha - \beta = \phi - \theta$, and $e\cos\frac{1}{2}(\alpha+\beta) = \cos\frac{1}{2}(\phi+\theta)$.

$$b = 1 - e\cos\alpha,$$

$$a = 1 - e\cos\beta,$$

$$c^2 = (\cos\alpha - \cos\beta)^2 + (1-e^2)(\sin\alpha - \sin\beta)^2;$$

$$\therefore\ ab\cos C = \tfrac{1}{2}(a^2 + b^2 - c^2)$$

$$= \cos(\alpha-\beta) - e(\cos\alpha + \cos\beta) + e^2(1 - \sin\alpha\sin\beta),$$

$$\text{and } ab = 1 - e(\cos\alpha + \cos\beta) + e^2\cos\alpha\cos\beta;$$

$$\therefore\ ab\cos^2\tfrac{1}{2}C = \{\cos\tfrac{1}{2}(\alpha-\beta) - e\cos\tfrac{1}{2}(\alpha+\beta)\}^2$$

$$= (1-\cos\phi)(1-\cos\theta) \quad (1),$$

$$\text{and } 2e(\cos\alpha + \cos\beta) = 4e\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$= 2(\cos\phi + \cos\theta);$$

$$\therefore\ \{\cos\tfrac{1}{2}(\alpha-\beta) + e\cos\tfrac{1}{2}(\alpha+\beta)\}^2 = (1+\cos\phi)(1+\cos\theta);$$

$$\therefore\ \cos\tfrac{1}{2}(\alpha-\beta) + e\cos\tfrac{1}{2}(\alpha+\beta) = 2\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\theta,$$

$$\cos\tfrac{1}{2}(\alpha-\beta) - e\cos\tfrac{1}{2}(\alpha+\beta) = 2\sin\tfrac{1}{2}\phi\sin\tfrac{1}{2}\theta,$$

from (1);

$$\therefore\ \cos\tfrac{1}{2}(\alpha-\beta) = \cos\tfrac{1}{2}(\phi-\theta), \text{ or } \alpha - \beta = \phi - \theta,$$

$$e\cos\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\phi+\theta).$$

62. Let O (fig. 50) be any point in the radius $AC = 1$, of a circle

$$AP = \frac{1}{n}\theta,\ PP_1 = P_1P_2 = \&c. = \frac{1}{n}2\pi, \text{ then}$$

$$OC^{2n} - 2\cos\theta\, OC^n + 1 = OP^2 . OP_1^2 . OP_2^2 \ldots OP_{n-1}^2.$$

For joining CP, CP_1, &c. we have $OP^2 = OC^2 - 2\,OC\cos\frac{1}{n}\theta + 1$,

$OP_1^2 = OC^2 - 2\,OC\cos\frac{1}{n}(2\pi+\theta) + 1$, &c., and therefore, (Theory of Equations, p. 27), the second member is the product of the quadratic factors of the first member.

63. To find the radii of three circles so inscribed in a triangle that each of them touches the two others and two sides of the triangle.

Let x, y, be the radii CN, $C'N'$, (fig. 51), angles A, B, C be 2α, 2β and 2γ, and radius of inscribed circle = 1; then $(NN')^2 = (x+y)^2 - (x-y)^2 = 4xy$; and equating two values of AB

$$x \cot \alpha + 2\sqrt{xy} + y \cot \beta = \cot \alpha + \cot \beta = \frac{\cos \gamma}{\sin \alpha \sin \beta} \text{ (1)},$$

since $\alpha + \beta + \gamma = \frac{1}{2}\pi$; and similarly if z be the radius of the third circle,

$$y \cot \beta + 2\sqrt{yz} + z \cot \gamma = \frac{\cos \alpha}{\sin \beta \sin \gamma},$$

$$z \cot \gamma + 2\sqrt{xz} + x \cot \alpha = \frac{\cos \beta}{\sin \alpha \sin \gamma};$$

$$\therefore\ x \cos^2 \alpha \sin \beta + 2 \sin \beta \sin \alpha \cos \alpha \sqrt{xy} + y \cos \beta \sin \alpha \cos \alpha = \cos \gamma \cos \alpha,$$

$$y \cos \beta \sin \gamma \cos \gamma + 2 \sin \beta \sin \gamma \cos \gamma \sqrt{yz} + z \cos^2 \gamma \sin \beta = \cos \alpha \cos \gamma;$$

equating these equals and observing that (Prob. 5)
$\cos \beta (\sin \alpha \cos \alpha - \sin \gamma \cos \gamma) = \sin \beta (\sin^2 \alpha - \sin^2 \gamma)$, we get

$$\frac{x \cos^2 \alpha + y \sin^2 \alpha + 2 \sin \alpha \cos \alpha \sqrt{xy}}{z \cos^2 \gamma + y \sin^2 \gamma + 2 \sin \gamma \cos \gamma \sqrt{zy}} = 1,$$

$$\therefore\ \sqrt{x} \cos \alpha + \sqrt{y} \sin \alpha = \sqrt{z} \cos \gamma + \sqrt{y} \sin \gamma;$$

similarly $\sqrt{z} \cos \gamma + \sqrt{x} \sin \gamma = \sqrt{y} \cos \beta + \sqrt{x} \sin \beta$,

$$\therefore\ \sqrt{x} (\cos \alpha + \sin \gamma - \sin \beta) = \sqrt{y} (\cos \beta + \sin \gamma - \sin \alpha);$$

$$\therefore\ \text{(Prob. 16)}\ \sqrt{\frac{y}{x}} = \frac{1 + \tan \frac{1}{2} \alpha}{1 + \tan \frac{1}{2} \beta} = \frac{1+p}{1+q}, \text{ suppose},$$

then $\cot \alpha = \dfrac{1-p^2}{2p}$, $\cot \beta = \dfrac{1-q^2}{2q}$; therefore substituting in (1),

$$x \left\{ \frac{1-p^2}{2p} + 2\frac{1+p}{1+q} + \left(\frac{1+p}{1+q}\right)^2 \frac{1-q^2}{2q} \right\} = \frac{(p+q)(1-pq)}{2pq};$$

$$\text{or } x\left\{1+\frac{p+q}{1-pq}\right\}=\frac{1+q}{1+p}, \quad \text{or } x\{1+\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\gamma)\}$$

$$=\frac{1+\tan\frac{1}{2}\beta}{1+\tan\frac{1}{2}\alpha};$$

$$\therefore\ x=\frac{1}{2}\,\frac{(1+\tan\frac{1}{2}\beta)(1+\tan\frac{1}{2}\gamma)}{1+\tan\frac{1}{2}\alpha}.$$

Similar expressions result for y and z.

64. If r be the radius of the circle inscribed in a triangle, and α, β, γ the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively,

$$\text{then } r=\sqrt{\alpha\beta}+\sqrt{\alpha\gamma}+\sqrt{\beta\gamma}.$$

It is easily seen that $\alpha = r-(r+\alpha)\sin\frac{1}{2}A$,

$$\therefore\ \frac{\alpha}{r}=\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}=\tan^2\tfrac{1}{4}(\pi-A);$$

$$\therefore\ \frac{1}{r}(\sqrt{\alpha\beta}+\sqrt{\alpha\gamma}+\sqrt{\beta\gamma})$$

$$=\tan\tfrac{1}{4}(\pi-A)\tan\tfrac{1}{4}(\pi-B)+\tan\tfrac{1}{4}(\pi-A)\tan\tfrac{1}{4}(\pi-C)$$
$$+\tan\tfrac{1}{4}(\pi-B)\tan\tfrac{1}{4}(\pi-C)=1,$$

$$\text{since}\quad \tfrac{1}{4}(\pi-A)+\tfrac{1}{4}(\pi-B)+\tfrac{1}{4}(\pi-C)=\tfrac{1}{2}\pi.$$

65. To sum the series

$$1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{2\,.\,3}-\frac{(n-5)(n-6)(n-7)}{2\,.\,3\,.\,4}+\&c.$$

If in the result of Art. 135, we make $2\cos\theta=1$, and therefore $\theta=\frac{1}{3}\pi$, we find the sum of this series to equal

$$\frac{1}{n}\left(1-2\cos\frac{n\pi}{3}\right).$$

66. To sum the series

$$a\sin\theta-\tfrac{1}{2}a^2\sin 2\theta+\tfrac{1}{3}a^3\sin 3\theta-\&c. \text{ in inf.}$$

Let the sum $= S$, and $2\sqrt{-1}\sin\theta = x-\frac{1}{x}$; then

$$2\sqrt{-1}\,S = a\left(x - \frac{1}{x}\right) - \tfrac{1}{2}a^2\left(x^2 - \frac{1}{x^2}\right) + \tfrac{1}{3}a^3\left(x^3 - \frac{1}{x^3}\right) - \&c.$$

$$= \log(1 + ax) - \log\left(1 + \frac{a}{x}\right) = \log\left(\frac{1 + ax}{1 + \frac{a}{x}}\right);$$

$$\therefore e^{2\sqrt{-1}S} = \frac{1 + ax}{1 + \frac{a}{x}}; \quad \therefore \frac{e^{2\sqrt{-1}S} - 1}{e^{2\sqrt{-1}S} + 1} = \frac{\left(x - \frac{1}{x}\right)}{2 + a\left(x + \frac{1}{x}\right)},$$

$$= \frac{2a\sqrt{-1}\sin\theta}{2 + 2a\cos\theta};$$

$$\therefore \tan S = \frac{a\sin\theta}{1 + a\cos\theta}, \quad \text{or } S = \tan^{-1}\left(\frac{a\sin\theta}{1 + a\cos\theta}\right).$$

67. Let $a = \sin\theta$, then

$$S = \tan^{-1}\left(\frac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right) = \cot^{-1}(1 + \cot\theta + \cot^2\theta).$$

68. Let $a = h\sin\theta$, then

$$S = \tan^{-1}\left(\frac{h\sin^2\theta}{1 + h\sin\theta\cos\theta}\right) = \tan^{-1}\frac{h}{1 + \cot^2\theta + h\cot\theta}$$

$$= \tan^{-1}\left\{\frac{h}{1 + x(x + h)}\right\} = \tan^{-1}(x + h) - \tan^{-1}x,$$

if $x = \cot\theta$, or $\tan^{-1}x = \frac{1}{2}\pi - \theta$;

$$\therefore \tan^{-1}(x + h) - \tan^{-1}x = h\sin\theta \,.\, \sin\theta - \tfrac{1}{2}h^2\sin^2\theta \,.\, \sin 2\theta + \tfrac{1}{3}h^3\sin^3\theta \,.\, \sin 3\theta - \&c.$$

69. To shew that

$$1 + \frac{n}{1}a\cos cx + \frac{n(n-1)}{1\,.\,2}a^2\cos 2cx + \&c. = r^n\cos n\theta,$$

$$\frac{n}{1}a\sin cx + \frac{n(n-1)}{1\,.\,2}a^2\sin 2cx + \&c. = r^n\sin n\theta,$$

where $\tan\theta = \dfrac{a\sin cx}{1 + a\cos cx}$, and $r^2 = 1 + 2a\cos cx + a^2$.

Let $2\cos cx = z + z^{-1}$;

$$\therefore e^{2\theta\sqrt{-1}} = \frac{1+az}{1+az^{-1}} = \frac{(1+az)^2}{r^2} = \frac{r^2}{(1+az^{-1})^2},$$

$$\therefore \tfrac{1}{2}(1+az)^n \pm \tfrac{1}{2}(1+az^{-1})^n = \tfrac{1}{2}r^n\left(e^{n\theta\sqrt{-1}} \pm e^{-n\theta\sqrt{-1}}\right);$$

$$\therefore 1 + \frac{n}{1}a\cos cx + \frac{n(n-1)}{1\,.\,2}a^2\cos 2cx + \&c. = r^n\cos n\theta,$$

$$\frac{n}{1}a\sin cx + \frac{n(n-1)}{1\,.\,2}a^2\sin 2cx + \&c. = r^n\sin n\theta.$$

70. To sum the series

$$1 + \frac{a}{1}\cos\theta + \frac{a^2}{1\,.\,2}\cos 2\theta + \frac{a^3}{1\,.\,2\,.\,3}\cos 3\theta + \&c.,$$

$$2S = 2 + \frac{a}{1}\left(x + \frac{1}{x}\right) + \frac{a^2}{1\,.\,2}\left(x^2 + \frac{1}{x^2}\right) + \&c.$$

$$= e^{ax} + e^{\frac{a}{x}} = e^{a(\cos\theta + \sqrt{-1}\sin\theta)} + e^{a(\cos\theta - \sqrt{-1}\sin\theta)}$$

$$= e^{a\cos\theta}\, 2\cos(a\sin\theta).$$

Similarly, $\frac{a}{1}\sin\theta + \frac{a^2}{1\,.\,2}\sin 2\theta + \frac{a^3}{1\,.\,2\,.\,3}\sin 3\theta + \&c.$

$$= e^{a\cos\theta}\sin(a\sin\theta).$$

71. If $\cot^{-1}(x-1) - \cot^{-1}(x+1) = \frac{\pi}{12}$,

prove that $x = \pm(\sqrt{3}+1)$.

72. In the equation $\tan 3\theta = n\tan\theta$, shew that n must equal $2+\sqrt{3}$, in order that θ may correspond to 15^0.

73. Find the logarithms of 8 and 9 having given $\log 6 = a$, $\log 15 = b$.

$$9 = \frac{6 \times 15}{10}, \quad \therefore \log 9 = a + b - 1,$$

$$8^2 = 4^3 = \left(\frac{60}{15}\right)^3; \quad \therefore \log 8 = \frac{3}{2}(a - b + 1).$$

Find log 83349, having given $\log 3 = a$, $\log 2.1 = b$.

$$83349 = 3^5 \times 7^3 = 3^2 \times (2.1)^3 \times 10^3;$$

$$\therefore \log 83349 = 2a + 3b + 3.$$

Find x in the equation $3^x = 15$, having given

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 47712125,$$

$$3^{x-1} = \frac{10}{2}, \quad \therefore (x-1)\log 3 = 1 - \log 2;$$

$$\therefore x = 1 + \frac{\cdot 69897}{\cdot 47712125} = 2.465.$$

Find x in logarithms from the equations $a^{b^x} = c$,

$$a^{x+m} - \frac{b}{a^{x+n}} = c;$$

and adapt $\frac{1}{4}\sqrt{\{(a^2+c^2)^2 - (a^2-c^2)^2 - (a^2-b^2+c^2)^2\}}$ to logarithmic computation; it is only another form of the expression for the area of a triangle.

74. Prove that $(\frac{1}{3})^{\frac{187}{14}} = \cdot 00000042366$, having given

$$\log 3 = \cdot 4771213, \text{ and } \log 4\cdot 2366 = \cdot 6270227.$$

75. To prove that $\frac{1}{4}\pi = \frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+\&c.}$.

$$\text{Let } f(x) = \frac{x^2}{n+}\,\frac{(x+n)^2}{n+}\,\frac{(x+2n)^2}{n+\&c.};$$

$$\therefore f(x) = \frac{x^2}{n + f(x+n)}.$$

$$\text{Assume } f(x) = \frac{1}{\phi(x)} - x, \quad \text{or } \phi(x) = \frac{1}{x + f(x)}.$$

$$\therefore \phi(x) + \phi(x+n) = \frac{1}{x},$$

which is satisfied by the value

$$\phi(x) = \frac{1}{x} - \frac{1}{x+n} + \frac{1}{x+2n} - \frac{1}{x+3n} + \&c.$$

Let $x = 1$, $n = 2$; then (Art. 145),

$$\phi(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. = \frac{1}{4}\pi;$$

$$\text{and } \phi(1) = \frac{1}{1 + f(1)} = \frac{1}{1+} \frac{1^2}{2+} \frac{3^2}{2 + \&c.};$$

$$\therefore \frac{1}{4}\pi = \frac{1}{1+} \frac{1^2}{2+} \frac{3^2}{2+} \frac{5^2}{2 + \&c.}.$$

76. To prove that $\tan z = \dfrac{z}{1-} \dfrac{z^2}{3-} \dfrac{z^2}{5 - \&c.}$.

Let $f(x) = \dfrac{n}{x+} \dfrac{n}{(x+1)+} \dfrac{n}{(x+2) + \&c.}$;

$$\therefore f(x) = \frac{n}{x + f(x+1)}.$$

$$\text{Assume } f(x) = \frac{n}{x} \frac{\phi(x+1)}{\phi(x)};$$

$$\therefore \phi(x) - \phi(x+1) = \frac{n}{x(x+1)} \phi(x+2),$$

which is satisfied by the value

$$\phi(x) = 1 + \frac{n}{x} + \frac{1}{2} \frac{n^2}{x(x+1)} + \frac{1}{2.3} \frac{n^3}{x(x+1)(x+2)} + \&c.$$

Let $x = \frac{1}{2}$, then (Art. 144.)

$$f(\tfrac{1}{2}) = 2n . \frac{1 + \dfrac{4n}{2.3} + \dfrac{16n^2}{\lfloor 5} + \&c.}{1 + \dfrac{4n}{1.2} + \dfrac{16n^2}{\lfloor 4} + \&c.} = \sqrt{n} . \frac{e^{2\sqrt{n}} - e^{-2\sqrt{n}}}{e^{2\sqrt{n}} + e^{-2\sqrt{n}}},$$

$$\text{and } f(\tfrac{1}{2}) = \frac{2n}{1+} \frac{4n}{3+} \frac{4n}{5 + \&c.};$$

hence, replacing $2\sqrt{n}$ by $z\sqrt{-1}$,

$$\tan z = \frac{z}{1-} \frac{z^2}{3-} \frac{z^2}{5 - \&c.}.$$

77. The ratio of the circumference of a circle to its diameter is incommensurable, that is, it cannot be expressed by any finite rational number either whole or fractional.

In order to prove this, it will be necessary to premise the following property of continued fractions.

If $S = \dfrac{m}{n - \dfrac{m'}{n' - \dfrac{m''}{n'' - \&c.}}}$, where m, m', m'', &c., n, n', &c. are whole numbers, and $\dfrac{m}{n}, \dfrac{m'}{n'}$, &c., each < 1 and their number infinite, S is incommensurable.

First S is < 1. For $n - \dfrac{m'}{n'} > n - 1$, and therefore $> m$, since m is an integer less than n. Similarly $n' - \dfrac{m''}{n''} > m'$. Hence $\dfrac{m}{n - \dfrac{m'}{n'}} < 1$, and $\dfrac{m}{n - \dfrac{m'}{n' - \dfrac{m''}{n''}}}$ is < 1. And thus we may shew that the whole value of the continued fraction is < 1. If therefore S be commensurable, let it $= \dfrac{B}{A}$, where A and B are integers, and let C, D, &c. be such that

$$\frac{C}{B} = \frac{m'}{n' - \dfrac{m''}{n'' - \&c.}}\quad \frac{D}{C} = \frac{m''}{n'' - \dfrac{m'''}{n''' - \&c.}}\ \&c.$$

then since each of these expressions is < 1, the series of terms A, B, C, D, &c. decreases continually. But since

$\dfrac{B}{A} = \dfrac{m}{n - \dfrac{C}{B}}$ we have $C = nB - mA$; similarly $D = n'C - m'B$, &c.; therefore since A and B are integers, the quantities C, D, &c. are also integers, and we have a decreasing series of integers continued indefinitely, which is absurd. Hence S must be incommensurable.

The same is true although the quantities $\dfrac{m}{n}, \dfrac{m'}{n'}$, &c., in the beginning of the series, are not < 1, provided they ultimately become so. For the value of the latter part of the continued fraction is incommensurable, as has been proved; and,

performing the operations indicated, S will also be incommensurable.

Now it has been proved (Prob. 76) that, m and n being any whole numbers,

$$\tan\frac{m}{n} = \frac{m}{n -}\,\frac{m^2}{3n -}\,\frac{m^2}{5n - \&c.},$$

which is incommensurable, since ultimately the quantity $\frac{m^2}{(2r-1)n}$ is < 1. If therefore $\frac{1}{4}\pi$ be commensurable, its tangent will be incommensurable; but it $= 1$, and therefore $\frac{1}{4}\pi$, and consequently π, is incommensurable.

78. To prove that, α, β, γ being any angles,

$$\sin(\alpha+\gamma) \,.\, \sin(\beta+\gamma) = \sin(\alpha+\beta+\gamma)\sin\gamma + \sin\alpha\sin\beta.$$

Multiplying by 2, this resolves itself into

$$\cos(\alpha-\beta) - \cos(\alpha+\beta+2\gamma) = \cos(\alpha+\beta) - \cos(\alpha+\beta+2\gamma) \\ + 2\sin\alpha\sin\beta,$$

which is evidently true.

79. $$1 + \sin\alpha\sin\beta + \sin\alpha\sin\gamma + \sin\beta\sin\gamma - \cos\alpha\cos\beta\cos\gamma \\ = 2\left\{\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}\gamma + \cos\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}\gamma\right\}^2.$$

Twice the second member when developed becomes

$$\{1 - \cos(\alpha+\beta)\}(1+\cos\gamma) + \{1 + \cos(\alpha-\beta)\} \\ \times (1 - \cos\gamma) + 2(\sin\alpha + \sin\beta)\sin\gamma,$$

which is identical with twice the first member.

80. $$\sin\alpha + \sin\beta + \sin\gamma = 4\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\beta+\gamma) \\ + \sin(\alpha+\beta+\gamma).$$

$$2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha+\gamma) = \cos\tfrac{1}{2}(\beta-\gamma) - \cos\tfrac{1}{2}(2\alpha+\beta+\gamma);$$

$$\therefore\ 4\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\beta+\gamma) = 2\sin\tfrac{1}{2}(\beta+\gamma) \\ \times \cos\tfrac{1}{2}(\beta-\gamma) - 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(2\alpha+\beta+\gamma) \\ = \sin\beta + \sin\gamma + \sin\alpha - \sin(\alpha+\beta+\gamma).$$

81. $$\cos\alpha + \cos\beta + \cos\gamma = 4\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\beta+\gamma) \\ - \cos(\alpha+\beta+\gamma).$$

82. $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma+\dfrac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}$.

The 1st member $=\dfrac{\sin\gamma}{\cos\gamma}+\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

$$=\frac{\{\cos\alpha\cos\beta-\cos(\alpha+\beta)\}\sin\gamma+\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$$

$$=\frac{\sin\alpha\sin\beta\sin\gamma+\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}.$$

83. $\cot\alpha+\cot\beta+\cot\gamma=\cot\alpha\cot\beta\cot\gamma$

$$-\frac{\cos(\alpha+\beta+\gamma)}{\sin\alpha\sin\beta\sin\gamma}.$$

84. $1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$

$=4\sin\frac{1}{2}(\alpha+\beta+\gamma)\sin\frac{1}{2}(\beta+\gamma-\alpha)\sin\frac{1}{2}(\alpha+\gamma-\beta)\sin\frac{1}{2}(\alpha+\beta-\gamma)$.

$2\sin\frac{1}{2}(\alpha+\beta+\gamma)\sin\frac{1}{2}(\beta+\gamma-\alpha)=\cos\alpha-\cos(\beta+\gamma)$

$2\sin\frac{1}{2}(\alpha+\gamma-\beta)\sin\frac{1}{2}(\alpha+\beta-\gamma)=\cos(\beta-\gamma)-\cos\alpha$;

$\therefore$ the 2nd member

$=-\cos^2\alpha-\cos(\beta+\gamma)\cos(\beta-\gamma)+\cos\alpha\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}$

$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$, (Art. 56).

Similarly,

$4\cos\frac{1}{2}(\alpha+\beta+\gamma)\cos\frac{1}{2}(\beta+\gamma-\alpha)\cos\frac{1}{2}(\alpha+\gamma-\beta)$

$\times\cos\frac{1}{2}(\alpha+\beta-\gamma)=-1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$.

85. To find the value of $\cos\alpha$, where $17\alpha=\pi$.

It may be shewn, as in Art. 153, that

$$\cos\alpha+\cos3\alpha+\cos5\alpha+\&c.+\cos15\alpha=\tfrac{1}{2}.$$

Let $p=\cos3\alpha+\cos5\alpha+\cos7\alpha+\cos11\alpha$,

$q=\cos\alpha+\cos9\alpha+\cos13\alpha+\cos15\alpha$;

then, multiplying, and replacing the product of every two cosines by the sum of two cosines, we get

$$pq=2\{\cos2\alpha+\cos4\alpha+\cos6\alpha+\&c.+\cos16\alpha\}$$
$$=-2\{\cos15\alpha+\cos13\alpha+\&c.+\cos\alpha\}=-1,$$

and $p+q=\frac{1}{2}$; hence p and q are known. Next let

$$\left.\begin{aligned} r &= \cos 3a + \cos 5a \\ s &= \cos 7a + \cos 11a \end{aligned}\right\} \quad \text{then } r + s = p, \quad rs = -\tfrac{1}{4},$$

$$\left.\begin{aligned} t &= \cos a + \cos 13a \\ u &= \cos 9a + \cos 15a \end{aligned}\right\} \quad t + u = q, \quad tu = -$$

hence r and t are known. But

$$\cos a + \cos 13a = t,$$

$$\begin{aligned} \cos a \,.\, \cos 13a &= \tfrac{1}{2}(\cos 12a + \cos 14a) \\ &= -\tfrac{1}{2}(\cos 5a + \cos 3a) = -\tfrac{1}{2}r, \end{aligned}$$

therefore $\cos a$ becomes known, by the solution of quadratic equations only. For the principle which directs the selection of the cosines to form p, q, &c. see Theory of Algebraical Equations, Art. 80.

86. A, B, C, D, &c. are the angular points of a regular hexagon inscribed in a circle radius $OA = r$; join AC cutting OB in P, join PD cutting OC in Q, join QE cutting OD in R, and so on; and shew that $OP = \frac{1}{2}r$, $OQ = \frac{1}{3}r$, $OR = \frac{1}{4}r$ and so on.

87. The sum of the squares of the sides of a quadrilateral figure is greater than the sum of the squares of its diagonals by four times the square of the line joining the middle points of the diagonals.

88. A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 55^0 with a horizontal line drawn through his eye; receding backwards 30 feet he then finds it to subtend an angle of 48^0. Required the breadth of the river.

$L \sin 7^0 = 9.08589$, $L \sin 35^0 = 9.75859$, $L \sin 48^0 = 9.87107$, $\log 3 = .47712$, $\log 1.0493 = .02089$.

89. The altitude of the least equilateral triangle that can circumscribe a given triangle $= \left\{a^2 + b^2 - 2ab\cos\left(\frac{1}{3}\pi + C\right)\right\}^{\frac{1}{2}}$.

A TREATISE

ON

SPHERICAL TRIGONOMETRY.

1. THE boundary of every plane section of a sphere is a circle.

If the cutting plane pass through the center, this is evident; and in this case the section is called a *great circle*, and is determined when any two points on the surface of the sphere through which it passes are given. All great circles are equal to one another, since they have the same radius, namely that of the sphere; and they all bisect one another, since their planes intersect in diameters of the sphere. Hence the distance of the points of intersection of two great circles measured on the sphere is a semi-circumference.

If the cutting plane does not pass through the center of the sphere, from O (fig. 1) the center, drop upon it the perpendicular OC, and join C with any point A in the boundary of the section; then

$$AC = \sqrt{AO^2 - OC^2}, \text{ which is invariable;}$$

therefore the boundary of the section is a circle whose center is C; and it is called a *small circle*.

Arcs of small circles are very rarely used; and when, hereafter, an arc of a circle is mentioned, an arc of a great circle, unless the contrary be specified, is invariably intended; and in most cases it is employed to denote the angle which it subtends at the center of the sphere, no regard being had to the radius of the sphere.

2. If OC be produced both ways to meet the surface of the sphere in P and P', then

$$PA = \sqrt{PC^2 + AC^2}, \text{ which is invariable.}$$

Also if PAM be an arc of a great circle passing through P and A, since in equal circles equal straight lines cut off

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equal arcs, the length of the arc PA is invariable. Therefore the distance of P is the same from every point in the perimeter of the circle AB, whether measured along the straight line, or the arc of a great circle, drawn from it to the point.

The point P, and the point P' which has evidently the same property, are called the nearer and more remote *poles* of the circle AB; being the extremities of that diameter of the sphere which is perpendicular to the plane of the circle. They are also the poles of all circles of the sphere whose planes are parallel to ACB.

If MN be the *great* circle of which P is the pole, since OP is perpendicular to the plane MON, and the angle POM is, consequently, a right angle, the distance of P from every point in the boundary of MN, measured on a great circle, is a quadrant.

3. The angle at which two arcs of great circles intersect on the surface of the sphere (in the same way as for any other curves) is the angle between their tangents at the point of intersection, and, consequently, is the same as the angle between the planes in which the arcs lie; for, as the tangents are situated in the same planes, respectively, with the arcs, and are perpendicular to the radius of the sphere which is the intersection of those planes, the angle between the tangents is the same as the angle contained between the planes.

Thus, let two arcs of great circles PA, PB, (fig. 1) intersect in P, and let two tangents be drawn to them, viz. PD which will be in the plane POA, and PE in the plane POB; then since PD, PE, are both perpendicular to PO, $\angle APB = \angle DPE =$ inclination of the planes in which the arcs are situated.

4. Let the arcs PA, PB, be produced to meet the great circle of which P is the pole in M, N, and any small circle of which P is the pole in A, B; and join OM, ON, CA, CB; then since PD, PE are respectively parallel to OM, ON, $\angle DPE = \angle MON$; and therefore

$$\angle APB = \angle MON = \text{arc } MN,$$

employing the arc, according to a preceding remark, to express the angle which it subtends at the center. This shews that if two arcs containing any angle be produced till each is a quadrant, the arc joining their extremities (which will be a portion of the great circle of which their point of intersection is the pole) will measure the angle they include.

Again, AC the radius of the small circle AB

$$= OA \sin AOP = OA \sin PA;$$

and length of arc $AB = AC \times$ circular measure of $\angle ACB$

$$= OA \sin PA \times \text{circular measure of } \angle MON$$

$$= \text{length of arc } MN \times \sin PA.$$

5. The planes of all great circles passing through P will contain OP, and therefore be perpendicular to the plane MON; therefore all great circles passing through P will cut the great circle MN, of which P is the pole, at right angles. Great circles which pass through the pole of another great circle are called *secondaries* to the latter.

Hence, a great circle MN being given, its pole P is determined either (1) by measuring an arc MP equal to a quadrant on any great circle perpendicular to MN; or (2) by drawing any two great circles, not in the same plane, at right angles to MN, and producing them till they intersect in P.

And, conversely, if a point on the sphere be such that an arc of a great circle drawn from it perpendicular to a proposed circle is a quadrant, or such that quadrants of two different great circles are intercepted between it and the proposed circle, then that point is the pole of the proposed circle.

6. The arc joining the poles of two great circles subtends an angle at the center equal to their inclination; and the point of intersection of the great circles (i. e. the extremity of the diameter in which their planes intersect) is the pole of the great circle in which their poles lie.

Let P and Q (fig. 2) be the poles of AC, BC, two great circles whose planes intersect in the diameter OC, then each of the angles POC, QOC, is a right angle, therefore CO is perpendicular to the plane POQ, and consequently C is the pole of the great circle $PQAB$ passing through P

and Q; and since each of the angles POA, QOB, is a right angle,

$$\angle POQ = \angle AOB = \angle ACB.$$

7. The portion of the surface of a sphere contained by three arcs of great circles which cut one another two and two, is called a *spherical triangle;* the planes in which the arcs lie forming a solid angle at the center of the sphere. The objects of investigation in Spherical Trigonometry are the relations subsisting between the angles at which the three plane faces containing a solid angle are inclined to one another, and the angles which the lines of intersection of those plane faces, or the three edges of the solid angle, form with one another.

Let O (fig. 3) be the vertex of a solid angle contained by the three plane faces AOB, BOC, COA, and let the arcs of great circles AB, BC, CA be the intersections of these planes with the surface of a sphere described from center O with any radius; then the inclinations of the planes AOB, BOC, COA, to one another, are identical with the angles A, B, C, of the spherical triangle; and the three angles at O are proportional to the sides AB, BC, CA. On this account the spherical triangle ABC, which is called the base of the solid angle at O, may be employed with great advantage in conducting the investigation of the relations of the angles of inclination of the faces and edges of the solid angle to one another; which relations, as has been said, are the proper objects of our research.

The sense in which the spherical triangle is employed being once understood, we may transfer our attention from the solid angle to the triangle in which its faces cut the sphere, and the solid angle need not be represented in our diagrams; but we must still keep in mind that, not the arcs forming the sides of the spherical triangle, but the *angles* which those arcs subtend at the center, are concerned in the calculations, so that the magnitude of the radius of the sphere is of no importance whatever: if it be supposed equal to unity then the sides become the circular measures of the angles which they subtend.

8. The three angles of a spherical triangle are usually

denoted by A, B, C; the *spherical angle* A, or the angle BAC contained by the arcs AB, AC, being, as explained above, the angle between the tangents to those arcs, or between the planes in which the arcs lie; and the angles which the sides BC, CA, AB, respectively opposite to A, B, C, subtend at the center of the sphere, are denoted by a, b, c. When, for the sake of brevity, the expression side BC, or side a, is used, the angle subtended at the center of the sphere by the arc BC which is opposite to the spherical angle A, is invariably meant.

9. Since, of the three plane angles which contain a solid angle, any one is less than two right angles, and less than the sum of the two others, and the sum of the three is less than four right angles; therefore, of the three arcs forming the sides of a spherical triangle, which are the measures of those angles, (1) any one is less than the semi-circumference, (2) any one is less than the sum of the two others, and (3) the sum of the three arcs is less than the circumference of a great circle.

Also if the great circle of which one side of a triangle is a portion, be completed, the hemisphere which it bounds will include the triangle; for if not, the points of intersection of two great circles would be separated by an arc greater than the semi-circumference.

It is easily seen that if sides greater than a semi-circumference were admitted, the same angular points might belong to different triangles; as, for instance, to the triangle formed by the arcs BC, CD, and BGD (fig. 5), as well as to the triangle $BCDH$; and that the solution of such triangles as $BCDG$ (if they should ever present themselves) can be made to depend on that of triangles limited to have no side greater than a semi-circumference.

10. If a spherical triangle be described on a sphere with the angular points of a given triangle for the poles of its sides, then the angular points of the triangle so described will be the poles of the sides of the given triangle; and its sides and angles will be respectively the supplements of the opposite angles and sides of the given triangle.

Let ABC (fig. 4) be a spherical triangle, $A'B'C'$ another

spherical triangle whose sides have the angular points of the former for their poles. Join $A'C$, $A'B$, by arcs of great circles, and produce AB, AC, on the surface of the sphere to meet $B'C'$ in G and H; then CA' is a quadrant, because C is the pole of $A'B'$, and BA' is a quadrant because B is the pole of $A'C'$; therefore (Art. 5) the great circle of which A' is the pole passes through B and C, or A' is the pole of BC; and similarly B' and C' may be shewn to be the poles of AC and AB. From this property, $A'B'C'$ is called, with respect to ABC, the *Polar* triangle. By construction, ABC is the polar triangle of $A'B'C'$.

$$\text{Also (Art. 4)} \ \angle A = HG = C'G - C'H = C'G + B'H - B'C'$$
$$= 180^0 - B'C',$$

since $C'G$, $B'H$ are each quadrants; that is, the angle A is the supplement of the angle subtended by $B'C'$ the side of the Polar triangle of which A is the pole, or which is opposite to A. Similarly it may be proved that $\angle B$, and $\angle C$, are the supplements of the angles subtended by $A'C'$, $A'B'$, the sides of the Polar triangle of which B and C are the poles. From this property the triangle $A'B'C'$ is sometimes called the *supplemental* triangle of ABC.

Hence, between the angles of the given triangle, and the sides a', b', c', of the supplemental triangle, we have the relations

$$A + a' = 180^0, \ B + b' = 180^0, \ C + c' = 180^0.$$

Also, since ABC is the supplemental triangle of $A'B'C'$,

$$\angle A' = 180^0 - BC, \quad \angle B' = 180^0 - AC, \quad \angle C' = 180^0 - AB;$$

so that between the sides of the given triangle and the angles of the supplemental triangle, we have the relations

$$A' + a = 180^0, \ B' + b = 180^0, \ C' + c = 180^0.$$

11. This shews that any solid angle contained by three planes being given, if through its vertex three planes be drawn perpendicular to its edges, these will form another solid angle whose edges are perpendicular to the faces of the former; and the inclinations of the faces and edges of the latter will be supplementary to the inclinations of the edges and faces, respectively, of the former.

12. The above Proposition is of great importance; for if a relation be proved to subsist among the angles and sides of any spherical triangle A, B, C, a, b, c, it will also hold for A', B', C', a', b', c', the angles and sides of the supplemental triangle, that is; for $180^0 - a$, $180^0 - b$, &c., Hence, any formula involving the sides and angles of a spherical triangle will still be true when, throughout it, for the angles the supplements of the sides are substituted, and for the sides the supplements of the angles.

13. Since if a', b', c', be the sides of the supplemental triangle,

$$A + B + C + a' + b' + c' = 6 \text{ right angles},$$

and $a' + b' + c'$ always lies between 0 and 4 right angles, therefore $A + B + C$, the sum of the angles, always lies between six right angles and two right angles.

A spherical triangle, therefore, different from a plane triangle, has not the sum of its angles invariable; nor can it have more than two of them $< 60^0$; but it may have two or all of them obtuse, or two or all of them right angles; in which latter case it will include one eighth of the surface of the sphere.

14. The area of a spherical triangle is the same fraction of the area of a hemisphere, that the excess of the sum of its three angles above two right angles is of 360^0.

Let ABC (fig. 5) be a spherical triangle; produce the arcs which contain its angles till they meet again two and two, which will happen when each has become equal to the semi-circumference. The triangle is now common to three different *lunes* (or portions of the spherical surface contained by two semi-circumferences) viz. $ABHDC$, $BCEGA$, and $CBFA$, the latter of which is equal to the sum of the triangles ABC and DCE, for DCE and ABF are evidently equal to one another, since they form the bases of vertically-opposite solid angles at O. Now the area of a lune is the same fraction of the area of the hemisphere (S), that the angle between the two semi-circumferences which contain it, is of 180^0; hence, by equating the two values of the area of each of the above-mentioned lunes, we have

$$\triangle ABC + BHDC = \frac{A}{180^0} S,$$

$$\text{lune } BCEGA = \frac{B}{180^0} S,$$

$$\triangle ABC + DCE = \frac{C}{180^0} S;$$

therefore by addition we get

$$2 \triangle ABC + S = \frac{A + B + C}{180^0} . S,$$

$$\text{or, area of triangle } ABC = \frac{A + B + C - 180^0}{360^0} . S.$$

The excess of the three angles above two right angles is called the *spherical excess;* the expression for the area cannot be negative, nor $> S$. (Art. 13.)

Relations between the sides and angles of a Spherical Triangle.

15. A solid angle such as has been before alluded to, has six elements, namely, the inclinations of the three plane faces to one another, and the inclinations of the three edges to one another; and when any three of these are given, the solid angle is, in general, completely determined. Similarly, a spherical triangle has six elements, namely, the three sides a, b, c, and the three angles A, B, C, respectively opposite to them; and it is, in general, completely determined when any three of these are given, as there exist relations between the three given elements and each of the unknown ones, by means of which the values of the unknown ones can be obtained.

Hence every formula of solution will involve four of the elements, and there can be only *four* distinct combinations of them; the first of three sides and an angle; the second of two sides and two angles respectively opposite to them; the third of two sides and two angles one of which is included by the sides; and the fourth of a side and three angles. We shall now investigate the formula belonging to each of these cases.

16. To find a relation between the three sides and any angle of a spherical triangle.

Let the tangent at A (fig. 3) to the arc AB meet OB produced (which is in the same plane with it) in D, and the tangent to the arc AC meet OC produced in E, and join DE. Then, equating the two values of the square of DE, obtained from the triangles DAE, DOE, we get

$$AD^2+AE^2-2AD.AE\cos DAE=OD^2+OE^2-2OD.OE\cos DOE;$$

therefore, observing that $OD^2-AD^2=OA^2$, $OE^2-AE^2=OA^2$, since the angles OAD, OAE, are right angles, and that $\angle DOE=BC=a$, $\angle DAE=BAC=A$, we find

$$2\,OD\,.\,OE\cos DOE=2\,OA^2+2\,AD\,.\,AE\cos DAE;$$

$$\therefore \cos DOE=\frac{OA}{OE}\cdot\frac{OA}{OD}+\frac{AE}{OE}\cdot\frac{AD}{OD}\cos DAE,$$

$$\text{or } \cos a=\cos b\cos c+\sin b\sin c\cos A,$$

which is the fundamental formula of Spherical Trigonometry.

17. The construction from which this result is obtained, fails unless the two sides b and c that contain the angle A are both less than 90^0, but puts no limitations upon the values of A and a. For the formula to be generally applicable to any angle of any spherical triangle whatever, the proof must be extended to the cases where b and c are, one or both, greater than 90^0, or, one or both, equal to 90^0.

First, suppose that only one of the containing sides b is greater than 90^0, and produce the sides CA, CB, (fig. 6) to intersect again in C'; then since AB, AC', are both less than 90^0, the formula is applicable to the angle $BAC'=180^0-A$ of the triangle BAC';

$$\therefore \cos(180^0-a)=\cos(180^0-b)\cos c$$
$$+\sin(180^0-b)\sin c\cos(180^0-A),$$

$$\text{or } \cos a=\cos b\cos c+\sin b\sin c\cos A,$$

the same result as would have been obtained if the formula had been applied immediately to the angle A of the triangle ABC.

Next, suppose that both the containing sides b and c are greater than 90^0, and produce them to intersect again in A' (fig. 7); then the formula is applicable to the angle $A' = A$ of the triangle $BA'C$;

$$\therefore \cos a = \cos(180^0 - b)\cos(180^0 - c) + \sin(180^0 - b)\sin(180^0 - c)\cos A,$$

$$\text{or } \cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which shews that the formula holds for an angle contained by two sides both greater than 90^0.

Again, suppose $AC = b = 90^0$ (fig. 7), and describe the arc DC from the point A as pole; then, provided $DC = A$ be different from 90^0, the formula is applicable to the angle D of the triangle BDC, and gives

$$\cos a = \cos(90^0 \sim c)\cos A + \sin(90^0 \sim c)\sin A \cos 90^0,$$

$$\text{or } \cos a = \sin c \cos A,$$

the same result as if the formula had been applied directly to the triangle ABC. If, in this case, DC is 90^0, then BC also is 90^0 (Art. 5), and the triangle is one with two quadrantal sides, and, consequently, with two right angles, for either of which the formula is evidently verified, as it leads to the equation $0 = 0$.

If both the containing sides b and c be 90^0, then $a = A$, which agrees with the result given by the formula, namely $\cos a = \cos A$.

18. If $a = b$, then $\cos A = \dfrac{\cos a(1 - \cos c)}{\sin a \sin c} = \cos B$;

$\therefore A = B$, since neither of them can exceed 180^0. Conversely, from the Polar Triangle it follows that if the angles at the base are equal, the sides opposite to them are equal.

Also if $A > B$, making $\angle DAB = \angle DBA$ (fig. 8), we have $DA = DB$, and consequently $CD + DB = CD + DA > AC$, or $a > b$; that is, the greater side is opposite to the greater angle.

19. If the formula, which we have just proved to express the relation between the three sides and any angle of any triangle, be applied in succession to the three angles A, B, C, it gives

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad (1),$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

three equations which comprehend implicitly the whole of Spherical Trigonometry; since, when any three elements are given, they enable us to find all the rest. As however it is desirable that each of the unknown elements should enter singly into the formula by which it is determined, we proceed to investigate the other three distinct formulæ which, in conjunction with (1), will always effect that object.

20. To find a relation between two sides and two angles respectively opposite to them.

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

$$\therefore \sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}}{\sin a \sin b \sin c},$$

taking the radical with a positive sign because $\sin A$ and $\sin a$ are both positive; now the second member is a symmetrical function of a, b, c, that is, an expression whose value remains unaltered when a, b, c are interchanged in any manner; consequently the ratio $\sin A \div \sin a$, has a constant value for each of the angles of the triangle; hence we have the three equations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

which shew, since this result is as general as the one from which it is deduced, that in any spherical triangle what-

ever, the sines of the angles are to one another as the sines of the sides opposite to them.

The direct method of proving this would be to obtain a relation between A, B, a, b, by deducing values of $\sin c$ and $\cos c$ from the two former of the equations of Art. 19, and substituting them in the formula $\sin^2 c + \cos^2 c = 1$; but the above is a simpler process.

21. The preceding proposition may be proved independently as follows.

Drop the perpendicular AP (fig. 3) upon the plane face BOC, and the perpendiculars PM, PN, upon the edges OB, OC, of the solid angle whose vertex is O and base the spherical triangle ABC, and join AM, AN. Then a line through P parallel to OM would be at right angles to the lines AP, PM, and therefore to the plane APM; consequently OM is at right angles to the plane AMP, and therefore $\angle AMP = \angle B$; similarly $\angle ANP = \angle C$.

$$\text{Hence} \quad AM \sin B = AP = AN \sin C;$$

$$\text{and} \quad AM = AO \sin AB, \quad AN = AO \sin AC,$$

$$\therefore \sin AB . \sin B = \sin AC . \sin C.$$

In the figure we have supposed the two sides AB, AC, to be both less than 90^0. Suppose only one of them AC (fig. 6) greater than 90^0, then from the triangle $AC'B$,

$$\sin c \sin (180^0 - B) = \sin (180^0 - b) \sin C',$$

$$\text{or} \quad \sin c \sin B = \sin b \sin C.$$

If both AB and AC be greater than 90^0, then from the triangle $A'BC$ (fig 7)

$$\sin (180^0 - c) \sin (180^0 - B) = \sin (180^0 - b) \sin (180^0 - C),$$

$$\text{or} \quad \sin c \sin B = \sin b \sin C.$$

If $AC = 90^0$, then from the triangle ADB, (fig. 6) C being the pole of AD, (unless $AB = 90^0$) $\sin AD \sin D = \sin AB \sin B$, or $\sin C = \sin c \sin B$. If AB also $= 90^0$, the triangle has two quadrantal sides, and consequently two right angles opposite to them, for which the formula is evidently verified. These results prove the relation above stated, to subsist for the angles, and sides opposite to them, of any triangle whatever.

22. To find a relation between two sides, and two angles one of which is included by the sides; for instance between a, b, A, C.

In the equation $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

substituting for $\cos c$ and $\sin c$ their values

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\sin c = \sin a \frac{\sin C}{\sin A},$$

we find

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin a \frac{\sin C}{\sin A} \cos A;$$

therefore, transposing $\cos a \cos^2 b$, and observing that

$$\cos a - \cos a \cos^2 b = \cos a \sin^2 b,$$

and dividing the whole by $\sin a \sin b$, we find

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.$$

This formula is as general as those from the combination of which it has been deduced; and the remembering of it may be facilitated by observing that each member begins with a cotangent multiplied by a sine, the first with any two sides taken at random, the second with two angles of which the former only is opposite to the former of the sides already involved; the last term is the product of the cosines of the elements whose sines are already involved.

23. To find a relation between the three angles and any side of a spherical triangle.

The simplest mode of effecting this is to apply the fundamental formula (1) to the supplemental triangle, which gives

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

then, substituting for a', b', c', A', their values in terms of the sides and angles of the proposed triangle, namely,

$$a' = 180^0 - A, \quad b' = 180^0 - B, \quad c' = 180^0 - C, \quad A' = 180^0 - a,$$

we find, for the side a, the formula

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a;$$

from which the corresponding formulæ for the sides b and c may be immediately deduced. The direct method of proof in this case would be to eliminate b and c from the three equations of Art. 19.

24. To prove Napier's Analogies.

We have by the preceding Article

$$\cos A + \cos B \cos C = \sin B \sin C \cos a,$$

$$\cos B + \cos A \cos C = \sin A \sin C \cos b,$$

$$\therefore \frac{\cos B + \cos A \cos C}{\cos A + \cos B \cos C} = \frac{\sin A \cos b}{\sin B \cos a} = \frac{\sin a \cos b}{\sin b \cos a};$$

therefore, comparing the difference of the terms of each ratio with the sum of the same terms, we get

$$\frac{\cos B - \cos A}{\cos B + \cos A} \cdot \frac{1 - \cos C}{1 + \cos C} = \frac{\sin (a - b)}{\sin (a + b)};$$

or $\tan \frac{1}{2} (A + B) \tan \frac{1}{2} (A - B) \,.\, \tan^2 \frac{1}{2} C$

$$= \frac{\sin \frac{1}{2} (a - b) \cos \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b) \cos \frac{1}{2} (a + b)} \quad (1).$$

$$\text{But } \frac{\sin A}{\sin B} = \frac{\sin a}{\sin b} \text{ gives } \frac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} (a + b)}{\tan \frac{1}{2} (a - b)}.$$

Multiplying these equations together, and dividing one by the other, and extracting the roots, we find

$$\tan \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \cot \tfrac{1}{2} C;$$

the positive signs being taken because $\frac{1}{2} (A - B)$ and $\frac{1}{2} (a - b)$ are less than 90^0 and of the same sign, and consequently from (1) $\tan (A + B)$ and $\cos \frac{1}{2} (a + b)$ must have the same sign; which shews that $\frac{1}{2} (A + B)$ and $\frac{1}{2} (a + b)$, since neither can exceed 180^0, are both less or both greater than 90^0.

Also, applying the formulæ just obtained to the polar triangle, or, which is the same thing, replacing A, B, C, a, b, by $180^0 - a$, $180^0 - b$, $180^0 - c$, $180^0 - A$, $180^0 - B$, we find

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

25. These formulæ, being under the form of analogies or proportions, are known by the name of Napier's Analogies; they are useful in the solution of triangles, the two former when two sides and the included angle are given, the two latter when one side and the two angles adjacent to it are given.

They may be also put into the following shape. Since $1 + \cos c = 1 + \cos a \cos b + \sin a \sin b\,(\cos^2 \frac{1}{2} C - \sin^2 \frac{1}{2} C)$

$$= \{1 + \cos (a-b)\} \cos^2 \tfrac{1}{2} C + \{1 + \cos (a+b)\} \sin^2 \tfrac{1}{2} C;$$

$$\therefore \cos^2 \tfrac{1}{2} c = \cos^2 \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2} C + \cos^2 \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2} C.$$

Similarly,

$$\sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2} C + \sin^2 \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2} C.$$

Now add unity to the square of each member of the two first analogies, and take account of the above values of $\cos \frac{1}{2} c$, $\sin \frac{1}{2} c$, and we find

$$\sec^2 \tfrac{1}{2}(A+B) = \frac{\cos^2 \frac{1}{2} c}{\cos^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C},$$

$$\sec^2 \tfrac{1}{2}(A-B) = \frac{\sin^2 \frac{1}{2} c}{\sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C};$$

or, extracting the roots, since $\frac{1}{2}(A+B)$, $\frac{1}{2}(a+b)$ are both greater or both less than 90^0,

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C,$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C;$$

and multiplying these respectively by the two first analogies,

$$\sin \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2} (a - b) \cos \tfrac{1}{2} C,$$

$$\sin \tfrac{1}{2} (A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2} (a - b) \cos \tfrac{1}{2} C.*$$

Solution of right-angled triangles.

26. When in the triangle ABC (fig. 10) the angle A is 90^0 that is, when the plane of the great circle AB is perpendicular to the plane of the great circle AC, the spherical triangle is called right-angled, and the side $BC = a$ is called the hypothenuse. By a right-angled triangle is usually understood a triangle with only one right angle, the two remaining angles being either acute or obtuse; for if there should be two right angles A and B, then the sides opposite to them BC and AC would be quadrants, and the angle C would equal AB; or if there should be three right angles, then all the sides would be quadrants (Art. 4); so that it is needless to consider these particular cases.

As a triangle is in general determined when any three parts are given, a right-angled triangle, with one exception, is determined when any two parts are given. To obtain the formulæ necessary for the solution of right-angled triangles, we have only to make $A = 90^0$ in the fundamental relations which have been investigated for any triangle whatever. These formulæ, by means of which, when any two parts of a right-angled triangle are given, the remaining three become known explicitly in terms of the given quantities, are embodied with equal elegance and convenience in Napier's Rules, the enunciation and proof of which are as follows.

27. The right angle being left out of consideration, the two sides including the right angle, and the complements of the hypothenuse and of the two other angles, are called *circular parts*. Any one of these being taken as the *middle* part, the two circular parts which, or whose complements, are immediately contiguous to it, or its complement, in position, are called the *adjacent* parts; and the other two parts are called the *opposite* parts. Then, whatever the middle part

* These four formulæ are sometimes proved by substituting in the developments of $\cos \frac{1}{2} (A + B)$, $\cos \frac{1}{2} (A - B)$, &c., the values of $\cos \frac{1}{2} A$, $\sin \frac{1}{2} A$, &c., in terms of the sides, found below, Art. 35.

be, whether a side, or the complement of the hypothenuse, or of an angle,

sine of the middle part

= product of the tangents of the adjacent parts;

sine of the middle part

= product of the cosines of the opposite parts.

1st. Let the complement of the hypothenuse $90^0 - a$ be middle part, then $90^0 - B$, $90^0 - C$, are the adjacent parts, and b, c, the opposite parts; and the formulæ

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

$$\cos a = \quad \cos b \cos c + \sin b \sin c \cos A,$$

since $A = 90^0$ and therefore $\cos A = 0$, $\sin A = 1$, become

$\cos a = \cot B \cot C$, or $\sin (90^0 - a) = \tan (90^0 - B) \tan (90^0 - C)$,

$\cos a = \cos b \cos c$, or $\sin (90^0 - a) = \cos b \cos c$.

2nd. Let the complement of an angle $90^0 - C$ be middle part, then $90^0 - a$ and b are the adjacent parts, and $90^0 - B$ and c the opposite parts; and the formulæ

$$\cot a \sin b = \cot A \sin C + \cos C \cos b,$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c,$$

become $\cos C = \cot a \tan b$, or $\sin (90^0 - C) = \tan (90^0 - a) \tan b$,

$\cos C = \sin B \cos c$, or $\sin (90^0 - C) = \cos (90^0 - B) \cos c$.

If $90^0 - B$ be taken for middle part, the rules may be proved exactly in the same way.

3rd. Let either side b be middle part, then $90^0 - C$ and c are the adjacent parts, and $90^0 - a$ and $90^0 - B$ the opposite parts, and the formulæ

$$\cot c \sin b = \cot C \sin A + \cos A \cos b,$$

$$\sin A \sin b = \sin a \sin B,$$

become $\sin b = \cot C \tan c$, or $\sin b = \tan (90^0 - C) \tan c$,

$\sin b = \sin a \sin B$, or $\sin b = \cos (90^0 - a) \cos (90^0 - B)$.

If c be taken for middle part, the rules may be proved exactly in the same way.

28. We are thus furnished with ten relations amongst the five parts of a right-angled triangle, each being a different combination of three of the parts; but five parts, taken three at a time, can be combined only in 10 different ways; consequently the above Rules, when any two parts whatever are given, will supply us with formulæ in which each of the remaining parts is *separately* combined with those two given parts, and in a form adapted for the immediate application of logarithms.

There can evidently be only six distinct cases, viz. 1 and 2 when the hypothenuse is given together with an angle, or with one of the sides containing the right angle; 3 and 4 when one of the sides containing the right angle is given together with the angle adjacent to it, or with the angle opposite to it; 5 and 6 when the two sides containing the right angle are given, or when the two angles are given. In applying Napier's Rules to obtain the three unknown parts from two given ones, it is sometimes requisite to take for middle part one of the given parts, and sometimes one of those that are sought, the sole object being to separately combine each of the unknown parts with the given ones. We shall now go through the six cases of solving right-angled triangles, first however premising the following observations.

29. The formula $\cos a = \cos b \cos c$ requires that either all three, or only one of the cosines should be positive; therefore the three sides of a right-angled triangle are either all less than 90^0, or two of them greater than 90^0, and the third less.

The formula $\sin c = \cot B \tan b = \tan b \div \tan B$ shews that $\tan B$ and $\tan b$ have the same sign, since $\sin c$ is always positive, c being less than 180^0; therefore B and b, since neither of them can exceed 180^0, are both greater or both less than 90^0; that is, in right-angled triangles a side and the angle opposite to it are either both greater or both less than a right angle; this is usually expressed by saying that they are of the same affection.

30. Hence (fig. 14) if DCD' be a great circle perpendicular to DAD', and CD be less than 90^0, CD is the least and CD' the greatest arc that can be intercepted between the point C and

DAD'; and of the arcs so intercepted the nearer any one is to CD, the less it is.

For since CD is less than 90^0, $\angle CBD < 90^0 < CDB$, and therefore $CD < CB$. Also $\angle CBD' > 90^0 > CD'B$, and therefore $CD' > CB$. And since $\angle CAB < 90^0$ and $\angle CBA > 90^0$, therefore $BC < AC$.

31. Case I. Having given the hypothenuse a, and an angle B, to find b, c, C.

Taking successively b, $90^0 - B$, and $90^0 - a$, for middle part, we get

$$\sin b = \sin a \sin B, \quad \cos B = \cot a \tan c, \quad \cos a = \cot B \cot C;$$

C and c are determined without ambiguity, and b must be of the same affection as B.

Case II. Having given the hypothenuse a, and a side b, to find c, B, C.

Taking successively $90^0 - a$, b, and $90^0 - C$ for middle part, we get

$$\cos a = \cos b \cos c, \quad \sin b = \sin a \sin B, \quad \cos C = \cot a \tan b.$$

By these formulæ c and C are determined without ambiguity, for there is only one angle less than 180^0 corresponding to a given cosine; also B, though determined by its sine, is not ambiguous since it must be of the same affection as b.

Case III. Having given one of the sides containing the right angle b, and the angle adjacent to it C, to find a, c, B.

Taking successively $90^0 - C$, b, and $90^0 - B$ for middle part, we get

$$\cos C = \tan b \cot a, \quad \sin b = \tan c \cot C, \quad \cos B = \cos b \sin C,$$

which determine a, c, B without ambiguity.

Case IV. Having given one of the sides containing the right angle b, and the angle opposite to it B, to find a, c, C.

Taking successively b, c, $90^0 - B$, for middle part, we get

$$\sin b = \sin B \sin a, \quad \sin c = \cot B \tan b, \quad \cos B = \sin C \cos b.$$

Here there is an ambiguity, all the unknown parts being determined by their sines; and it is easily seen that such ought to be the case. For if we produce the two sides BA, BC

(fig. 11) till they intersect again in B', we have a second right-angled triangle CAB' in which the side b and the angle $B' = B$ are evidently the same as in the triangle ABC; and the other parts of the second triangle are the supplements of the corresponding parts C, a, c of the first triangle. Hence we may take a either less or greater than 90^0, but having made the choice, c will be given by the relation $\cos a = \cos b \cos c$, and then C will be of the same affection as c; and thus the two triangles, which equally solve the problem, will be determined.

There will however be only one triangle if $b = B$, having two right angles; and none at all, if $\sin b > \sin B$, for then the value of $\sin a$ is impossible. In the latter case, since B and b are of the same affection, b is greater or less than B, according as B is acute or obtuse; and therefore B and b cannot form parts of a right-angled spherical triangle; for if P be the pole of AB, and B of DE, then when B is acute so that ABC is the triangle, $PC < PE$ (Art. 30), and therefore $AC < DE$, i.e. b cannot be greater than B; and when B is obtuse so that ABC' is the triangle, then $PC' > PE'$ and therefore $AC' > DE'$, i.e. b cannot be less than B.

Hence when $\angle B$ is acute, there is no triangle if $b > B$, one if $b = B$, and two if $b < B$; and when $\angle B$ is obtuse, there is no triangle if $b < B$, one if $b = B$, and two if $b > B$.

Case V. Having given the two sides containing the right angle b and c, to find a, B, C.

Taking successively $90^0 - a$, c, b, for middle part, we get

$$\cos a = \cos b \cos c, \quad \sin c = \cot B \tan b, \quad \sin b = \cot C \tan c,$$

which determine a, B, C, without ambiguity.

Case VI. Having given the two oblique angles B and C, to find a, b, c.

Taking successively $90^0 - a$, $90^0 - B$, $90^0 - C$ for middle part, we get

$$\cos a = \cot B \cot C, \quad \cos B = \cos b \sin C, \quad \cos C = \cos c \sin B.$$

These formulæ leave no ambiguity; and if the triangle is impossible they will shew it.

Solution of quadrantal and isosceles triangles.

32. Of other triangles which may be solved as right-angled triangles, the principal class is that called *quadrantal* triangles, in which one side, a, is a quadrant. Since the polar triangle in this case will have one angle $A' = 180^0 - a = 90^0$, applying Napier's Rules to it as in Art. 27, we get

$$\cos a' = \cot B' \cot C' \text{ or } = \cos b' \cos c',$$
$$\cos C' = \cot a' \tan b' \text{ or } = \sin B' \cos c',$$
$$\sin b' = \cot C' \tan c' \text{ or } = \sin a' \sin B'.$$

Therefore, substituting for a', b', &c. their values, we get

$$-\cos A = \cot b \cot c \text{ or } = \cos B \cos C,$$
$$-\cos c = \cot A \tan B \text{ or } = -\sin b \cos C,$$
$$\sin B = \cot c \tan C \text{ or } = \sin A \sin b;$$

or,
$$\sin (A-90^0) = \tan (90^0-b) \tan (90^0-c) \text{ or } = \cos B \cos C,$$
$$\sin (90^0-c) = \tan (A-90^0) \tan B \text{ or } = \cos (90^0-b) \cos C,$$
$$\sin B = \tan (90^0-c) \tan C \text{ or } = \cos (A-90^0) \cos (90^0-b);$$

which shew that if the complements of the two sides, the complement of the hypothenusal angle taken negatively, and the two other angles be taken for the circular parts, that is,

$$90^0 - b,\ 90^0 - c,\ -(90^0 - A),\ B \text{ and } C,$$

the quadrantal triangle may be at once solved by Napier's Rules.

33. If a triangle be isosceles, joining the vertex with the middle of the base by the arc of a great circle, we divide it into two right-angled triangles equal in all respects; if therefore any two parts of an isosceles triangle be given, (counting however the two equal sides as only one element and the two equal angles opposite to them only as one element,) all the parts of the triangle may be determined by the Rules for right-angled triangles.

Also, if in a spherical triangle the sum of any two sides $a + b = 180^0$, producing b and c till they intersect again in A' (fig. 7) we have $b + CA' = 180^0$; $\therefore$ $CB = A'C$; hence the solution of the triangle ABC is reduced to that of the isosceles triangle $A'CB$. The condition $a + b = 180^0$ is evidently the same as $A + B = 180^0$; for since $A'C = CB$, $\angle\ CA'B = CBA' = A$, therefore $A + B = CBA' + CBA = 180^0$.

Solution of oblique-angled triangles.

34. Oblique-angled triangles, in which there must always be given three of the six elements A, B, C, a, b, c, present six distinct cases, the data in them being as follows: 1. three sides; 2. two sides and an angle opposite to one of them; 3. two sides and the included angle; 4. two angles and the side opposite to one of them; 5. two angles and the side adjacent to them both; 6. three angles. All these cases are readily solved by means of the four fundamental relations expressed by the formulæ of Arts. 19—23; but as these formulæ require certain modifications to make them suitable for actual computation by means of logarithms, it is necessary to go through each case separately.

35. Case I. Having given the three sides, to find the angles.

We have (Art. 16) $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$,

which gives A, the second member being entirely known; and by similar formulæ may B and C be determined; but this formula is not suited to logarithmic calculation, a defect in it that may be supplied as follows. First we have

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A,$$

and substituting for $\cos A$ its value, we get successively

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b-c) - \cos a}{\sin b \sin c} = \frac{2 \sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}.$$

To simplify this, let half the perimeter of the triangle be denoted by s, so that $a+b+c=2s$; $\therefore a+b-c=2(s-c)$, $a-b+c=2(s-b)$; hence, substituting and extracting the root, we find

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}.$$

With equal facility may similar expressions for $\cos \frac{1}{2} A$ and $\tan \frac{1}{2} A$ be found. For

$$2\cos^2 \tfrac{1}{2}A = 1+\cos A = 1+\frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos(b+c)}{\sin b \sin c},$$

$$= \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c} = \frac{2\sin s \sin(s-a)}{\sin b \sin c},$$

$$\therefore \cos\tfrac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

Also $\tan\frac{1}{2}A = \sin\frac{1}{2}A \div \cos\frac{1}{2}A = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$.

These formulæ determine the angle without ambiguity; for since A is the angle of a spherical triangle, $\frac{1}{2}A$ is less than 90°. The principles which must guide us in selecting a formula for any particular case, are the same as those explained in Plane Trigonometry, Art. 113.

36. Taking twice the product of the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$, we find

$$\sin A = \frac{2}{\sin b \sin c}\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)};$$

which, as it requires seven logarithms, is not an advantageous formula for the calculation of A. It is the form to which the expression for $\sin A$ in Art. 20 may be reduced.

37. Case II. Having given two sides a, b, and the angle A opposite to one of them, to find c, B, C.

The angle B opposite to the side b may first of all be obtained from the relation

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

Then C and c may be determined by Napier's Analogies, which give

$$\tan\tfrac{1}{2}C = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}(A+B),$$

$$\tan\tfrac{1}{2}c = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a+b).$$

38. As the angle B is determined by its sine, it may be either greater or less than 90^0; this case, consequently, will often admit of two solutions; but for certain values of the given elements a, b, A there will be only one triangle, or none at all, under the usual restriction of having no side greater than 180^0. To the examination of these circumstances, analogous to what happens in the second case of Plane Triangles (Art. 105), we shall afterwards recur.

The ambiguity to a certain extent may be removed by observing that, as $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of the same affection (Art. 24), if $a+b>180^0$, then $A+B>180^0$; if therefore A be less than 90^0, B must be greater than 90^0, and consequently there can be only one triangle having the given elements, although there may be none; but if A be greater than 90^0, then a value of B either greater or less than 90^0 will satisfy the condition $A+B>180^0$, and consequently there may be two triangles having the given elements, although there may be none. Again, if $a+b<180^0$, then $A+B<180^0$; if therefore A be greater than 90^0, B must be less than 90^0, and there can be only one solution; but if A be less than 90^0, then a value of B either greater or less than 90^0 will satisfy the condition $A+B<180^0$, and there may be two solutions.

39. We may determine C and c directly from the given quantities, without first finding B. From Art. 22, we get

$$\cot a \sin b = \cot A \sin C + \cos b \cos C = \cos b \left(\cos C + \sin C \frac{\cot A}{\cos b}\right).$$

Let ϕ be a subsidiary angle determined by the equation

$$\tan\phi = \frac{\cot A}{\cos b};$$

$$\therefore \cot a \tan b = \frac{\cos(C-\phi)}{\cos\phi}, \text{ or } \cos(C-\phi) = \frac{\tan b \cos\phi}{\tan a},$$

from which $C-\phi$, and consequently C, may be obtained.

Again, we have $\cos a = \cos b \cos c + \sin b \sin c \cos A$

$$= \cos b\,(\cos c + \sin c \tan b \cos A).$$

Let θ be a subsidiary angle determined by the equation $\tan\theta = \tan b \cos A$;

$$\therefore \cos a = \cos b \cdot \frac{\cos(c-\theta)}{\cos\theta}, \text{ or } \cos(c-\theta) = \frac{\cos a \cos\theta}{\cos b},$$

from which $c - \theta$, and consequently c, may be obtained.

40. With respect to such transformations as those just made, and which are of constant occurrence in this subject, it may be observed that their object in every case, is to change into a product a binomial of the form $m \sin a + n \cos a$; this is done by first making one of the quantities m or n a factor of the whole expression, so that it becomes $m\left(\sin a + \frac{n}{m}\cos a\right)$, and then equating $\frac{n}{m}$ to the tangent or cotangent of an angle (ϕ), which is always allowable as the tangent and cotangent are susceptible of all values, by this substitution the expression is reduced to

$$\frac{m \sin(a+\phi)}{\cos\phi} \text{ or } \frac{m\cos(a-\phi)}{\sin\phi}.$$

41. It is also worth while remarking that the introduction of the subsidiary angles ϕ and θ amounts to dividing the proposed triangle into two right-angled triangles.

Thus, in the present case, if $CD = h$ (fig. 13) be an arc of a great circle drawn from the angle included by the given sides perpendicular to the opposite side, and if $\angle ACD = \phi$ and the segment $AD = \theta$, then from the right-angled triangles ACD, BCD, by Napier's Rules, we get

$$\cos b = \cot\phi \cot A, \quad \cos A = \cot b \tan\theta,$$

$$\cos(C-\phi) = \tan h \cot a = \frac{\cos\phi}{\cot b} \cdot \cot a,$$

$$\cos a = \cos h \cos(c-\theta) = \frac{\cos b}{\cos\theta} \cdot \cos(c-\theta),$$

which are identical with the former results, and involve the same ambiguities.

42. Case III. Having given two sides a, b, and the included angle C, to find A, B, c.

By Napier's Analogies we have

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C,$$

which determine $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, and consequently A and B; and these being known, we obtain c from the equation, $\sin c = \frac{\sin C}{\sin A} \sin a$, observing that the greater side is opposite to the greater angle. Or, c may be found by one of the formulæ of Art. 25, and then only two additional logarithms will be wanted.

43. If it be required to determine c independently of A and B, we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C = \cos a (\cos b + \sin b \tan a \cos C).$$

Let θ be a subsidiary angle determined by the equation $\tan \theta = \tan a \cos C$;

$$\therefore \cos c = \frac{\cos a \cos (b-\theta)}{\cos \theta},$$

which gives c without ambiguity.

Also A may be independently determined from the formula

$$\cot A = \frac{\cot a \sin b - \cos b \cos C}{\sin C} = \cot C \left(\frac{\cot a}{\cos C} \sin b - \cos b \right),$$

which, upon introducing the same subsidiary angle θ, becomes

$$\cot A = \frac{\cot C \sin (b-\theta)}{\sin \theta}.$$

It is easily seen that these results may be obtained by dropping the arc BD' perpendicular to AC, and calling the segment $CD' = \theta$. (fig. 13.)

The side c and the angle B may, in a similar manner, be directly found from the given quantities, by dropping a perpendicular from the angle A upon the opposite side.

44. Case IV. Having given two angles A and B, with the side adjacent to both c, to find a, b, C.

By Napier's Analogies we have

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

which determine $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$, and consequently a and b; and these being known, we have C from the equation

$$\sin C = \frac{\sin c}{\sin a} \sin A,$$

observing that the greater side is opposite the greater angle.

45. If it be required to determine C independently of a and b, we have

$$\begin{aligned} \cos C &= -\cos A \cos B + \sin A \sin B \cos c \\ &= \cos A(-\cos B + \sin B \tan A \cos c); \end{aligned}$$

and determining the subsidiary angle ϕ by the equation

$$\cot \phi = \tan A \cos c, \quad \text{we get } \cos C = \frac{\cos A \sin (B-\phi)}{\sin \phi}.$$

Also, a may be independently determined from the formula

$$\cot a \sin c = \cot A \sin B + \cos c \cos B = \cos c \left(\cos B + \sin B \frac{\cot A}{\cos c} \right);$$

which, upon introducing the same subsidiary angle ϕ, becomes

$$\cot a = \frac{\cot c \cos (B-\phi)}{\cos \phi}.$$

It is easily seen that these results may be obtained by dropping the arc BD' perpendicular to AC and calling $\angle ABD' = \phi$.

The side b and $\angle C$ may, in a similar way, be determined directly from the given quantities, by dropping the perpendicular from the angle A. This case is analogous to the third, and gives rise to no ambiguity.

46. Case V. Having given two angles A, B, and the side a opposite to one of them, to find b, c, C.

This case, being exactly analogous to the second, is treated in the same way, and gives rise to the same ambiguities.

The side b may be first of all obtained from the formula $\sin b = \dfrac{\sin B}{\sin A} \sin a$; and then C and c from the formulæ

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b).$$

47. We may also determine C and c directly from the given quantities; for we have

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$= \cos B(-\cos C + \sin C \tan B \cos a) = \frac{\cos B \sin (C - \psi)}{\sin \psi}.$$

ψ being a subsidiary angle determined from the equation $\cot \psi = \tan B \cos a$. Also the formula

$$\cot a \sin c = \cot A \sin B + \cos c \cos B,$$

if we determine ϕ by the equation $\cot \phi = \dfrac{\cot a}{\cos B}$, becomes

$$\cot A \sin B = \cos B \left(-\cos c + \sin c \frac{\cot a}{\cos B}\right) = \frac{\cos B \sin (c - \phi)}{\sin \phi}.$$

These results may evidently be obtained by dropping the arc CD perpendicular on AB, and calling $\angle BCD = \psi$, and the segment $BD = \phi$.

48. Case VI. Having given the three angles A, B, C, to find the sides a, b, c.

The formulæ of solution in this case are obtained by a process exactly similar to that in Case I. We have

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C};$$

$$2\sin^2\tfrac{1}{2}a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C} = \frac{\cos A + \cos(B+C)}{\sin B\sin C}$$

$$= -\frac{2\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B\sin C}.$$

Let $A+B+C=2S$; $\therefore B+C-A=2(S-A)$; hence, substituting and extracting the root, we get

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}.$$

Similarly, $\cos\frac{1}{2}a = \sqrt{\dfrac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}$,

$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

These values, which might have been derived from the formulæ of Case I. by means of the Polar Triangle, are always real. In the first place, S is always greater than 90^0 and less than 270^0, therefore $\cos S$ is always negative: also because in the supplementary triangle, $a' < b' + c'$,

$$\therefore 180^0 - A < 180^0 - B + 180^0 - C, \text{ or } B + C - A < 180^0;$$

consequently $S - A < 90^0$, and its cosine is positive; and in like manner $\cos(S-B)$, $\cos(S-C)$, are positive.

Ambiguous cases in the solution of oblique-angled triangles.

49. It appears that the only Cases in which any doubt can arise as to the values of the computed elements, are the second and the fifth; and it becomes necessary to investigate the conditions to which the given elements are respectively subject, when they correspond to two triangles, or to only one, or to none at all.

Let ADD', DCD', (fig. 14) be two great circles at right angles to one another, and ACE a circle making an acute angle CAD with ADD'; then (Art. 30) CD is less than 90^0 and is the least arc, and CD' is the greatest arc, that can be intercepted between C and the circle ADD'; and of the arcs so intercepted, those which are equally inclined to CD or CD' are equal to one another, and the nearer any arc is to CD the less it is, and the nearer any arc is to CD' the greater it is.

Suppose now the given angle A, according as it is acute or obtuse, to be represented by CAD or CAD', and let AC represent the given adjacent side b; then if the given opposite side a be intermediate in magnitude to the perpendiculars CD and CD', the small circle described with pole C and angular radius $= a$, will always cut ADD' in two points, and determine two admissible triangles having the proposed elements, A, b, a; except in the cases in which one or both of the triangles become inadmissible on account of having a side $> 180^0$, or an angle $= 180^0 - A$ instead of A.

But if the side a be less than CD, or greater than CD', the small circle described from C as pole with angular radius $= a$, will never intersect ADD', but fall entirely above it in the former case, and below it in the latter; and the construction of a triangle with the proposed elements will be impossible. This also appears from the formula; for if $a < CD$, then since $CD < 90^0$, $\sin a < \sin CD < \sin A \sin b$; and if $a > CD'$, then since $CD' > 90^0$, $\sin a < \sin CD' < \sin A \sin b$; so that in both cases the equation $\sin B = \sin A \sin b \div \sin a$ for determining B, is impossible.

Excluding therefore these impossible cases, and always supposing the circles to cut one another, we shall now examine in what cases one or both of the triangles determined by their intersections, are inadmissible on account of having a side $> 180^0$, or an angle $= 180^0 - A$ instead of A.

50. Case I. Let the given angle A be less than 90^0; and first suppose b less than 90^0, then AD must be also less than 90^0 (Art. 29) and consequently less than DE. If therefore a be less than b, it is evident that we may place an arc $BC = a$ between AC and CD, and another $CB' = a$ between FC and CD, and so construct two triangles that have the given elements; but if $a = b$ the triangle ABC disappears, and if $a > b$ the triangle ABC has the angle $180^0 - A$ instead of A, so that there only remains one triangle ACB' with the proposed elements, and that, only so long as $a < 180^0 - b$.

Next suppose AC* or $b > 90^0$, then if $a < 180^0 - b$ we may place $CB = CB' = a$ on each side of the perpendicular CD,

* The reader is requested to supply the letters in this part of the diagram.

and so construct two triangles that have the given elements; but if $a = \pi - b$ the triangle ACB' becomes a lune, and if $a > \pi - b$ the triangle ACB' has a side $> 180^0$, so that there only remains one triangle ACB that satisfies the conditions, and that, only so long as $a < b$.

Hence we conclude that there are, if

$$b < 90^0 \begin{cases} a < b & \text{two solutions,} \\ a = \text{or} > b & \text{one solution,} \\ a + b = \text{or} > 180^0 & \text{no solution;} \end{cases}$$

$$b > 90^0 \begin{cases} a < 180^0 - b & \text{two solutions,} \\ a = \text{or} > 180^0 - b & \text{one solution,} \\ a = \text{or} > b & \text{no solution.} \end{cases}$$

51. Case II. Let the given angle be greater than 90^0; then in exactly the same way, using the triangle ACD', instead of ACD, we may shew that there are, if

$$b < 90^0 \begin{cases} a > 180^0 - b & \text{two solutions,} \\ a = \text{or} < 180^0 - b & \text{one solution,} \\ a = \text{or} < b & \text{no solution;} \end{cases}$$

$$b > 90^0 \begin{cases} a > b & \text{two solutions,} \\ a = \text{or} < b & \text{one solution,} \\ a = \text{or} < 180^0 - b & \text{no solution.} \end{cases}$$

52. It is evident that by means of the Polar Triangle, the discussion of the 5th Case of the solution of oblique-angled triangles, in which the data are A, B, a, may be reduced to the above; and we may apply the results just obtained to that case, if we change a, b, A into A, B, a, and the sign $>$ into $<$, and $<$ into $>$. In those cases where the given elements correspond to only one solution, the calculation will still indicate two; but the one to be taken may be always discerned by means of the property that the greater side is opposite the greater angle.

Approximate solution of spherical triangles in certain cases.

53. We shall now give a few instances of the application of Spherical Trigonometry to cases which allow the exact for-

mulæ hitherto obtained, to be advantageously replaced by approximate ones of much greater simplicity. These instances chiefly occur in investigations having for their object the correct representation of a portion of the Earth's surface which is too large in extent to be considered as situated in one plane.

54. Let α, β, γ be the arcs forming the sides of a spherical triangle situated on the surface of the terrestrial globe whose radius we shall denote by r. Now although the arcs α, β, γ may be several miles in length, they are so small compared with r, that the angles subtended by them at the Earth's center are very small, usually considerably below 1^0; and for angles of that magnitude, the logarithmic tables do not enable us to attain sufficient exactness. It consequently is attended with much less trouble, and with equal accuracy, to reduce the solution of such triangles as we have described, to that of plane triangles, which may be effected by means of the following Proposition.

55. If the arcs which form the sides of a spherical triangle be very small relative to the radius of the sphere, then each of its angles will exceed the corresponding angle of the plane triangle whose sides are of the same length as the arcs forming the sides, by one-third of the spherical excess.

Let A, B, C denote the circular measures of the angles of the spherical triangle whose sides are the arcs α, β, γ; and A', B', C' the circular measures of the angles of the plane triangle whose sides are of the same length as α, β, γ;

$$\text{then } \cos A = \frac{\cos\frac{\alpha}{r} - \cos\frac{\beta}{r}\cos\frac{\gamma}{r}}{\sin\frac{\beta}{r}\sin\frac{\gamma}{r}};$$

$$\text{but } \cos\frac{\alpha}{r} = 1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{2.3.4r^4},$$

$$\sin\frac{\alpha}{r} = \frac{\alpha}{r} - \frac{\alpha^3}{2.3r^3},$$

expanding the cosine and sine in terms of the circular measure of the angle, and neglecting powers above the fourth; and similarly for $\frac{\beta}{r}$, $\frac{\gamma}{r}$;

$$\therefore \cos A = \frac{\frac{1}{2r^2}(\beta^2+\gamma^2-\alpha^2)+\frac{1}{24r^4}(\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2)}{\frac{\beta\gamma}{r^2}\left(1-\frac{\beta^2+\gamma^2}{6r^2}\right)}$$

$$=\frac{1}{2\beta\gamma}\left\{\beta^2+\gamma^2-\alpha^2+\frac{1}{12r^2}(\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2)\right\}\left\{1+\frac{\beta^2+\gamma^2}{6r^2}\right\}$$

$$=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\beta^2\gamma^2}{24\beta\gamma r^2}$$

$$=\cos A'-\frac{\beta\gamma}{6r^2}\sin^2 A'.$$

Let $A = A' + \theta$; $\therefore \cos A = \cos A' - \theta \sin A'$;

$$\therefore \theta = \frac{\beta\gamma \sin A'}{6r^2} = \frac{S}{3r^2}, \text{ or } A = A' + \frac{S}{3r^2},$$

if S denote the area of the plane triangle; hence, since S does not alter when A is interchanged with B or with C,

$$B = B' + \frac{S}{3r^2}, \quad C = C' + \frac{S}{3r^2};$$

$$\therefore A + B + C = \pi + \frac{S}{r^2} = \pi + E; \quad \therefore \frac{S}{r^2} = E,$$

and $A = A' + \frac{1}{3}E$, $\quad B = B' + \frac{1}{3}E$, $\quad C = C' + \frac{1}{3}E$.

56. Hence when three parts of the spherical triangle are given, we can always obtain three parts of the plane triangle; and by means of these all the parts of the spherical triangle will become known.

Suppose that the three arcs α, β, γ are given, then S the area of the plane triangle can be found at once, and therefore $E = \frac{S}{r^2 \sin 1''}$; if then the angles A', B', C' be computed in degrees, &c. from α, β, γ, we shall, by adding to each one

third of E expressed as above in seconds, obtain the values of the angles A, B, C.

Next suppose that A, β, γ are given, then $S = \frac{1}{2}\beta\gamma \sin A' = \frac{1}{2}\beta\gamma \sin A$ nearly, therefore $E = \frac{S}{r^2 \sin 1''}$ is known; hence in the plane triangle, β, γ, and $A' = A - \frac{1}{3}E$, are known, and consequently its remaining parts, and therefore those of the spherical triangle, may be found.

If A, a, β are given, then

$$\sin B' = \frac{\beta \sin A'}{a} = \frac{\beta \sin A}{a} \text{ nearly; therefore } C' = 180^\circ - A - B'$$

nearly; therefore $S = \frac{1}{2}\, a\beta \sin C'$, and $E = \frac{S}{r^2 \sin 1''}$.

If A, B, γ are given, then $S = \frac{\gamma^2 \sin A \sin B}{2 \sin (A + B)}$ nearly, and $E = \frac{S}{r^2 \sin 1''}$; and if A, B, a are given, then $C = C' = 180^\circ - A - B$ nearly, and we must proceed, as above, with the elements C, B, a.

57. To reduce an angle to the horizon.

Let BDC (fig. 12) be an angle situated in an inclined plane, and having its vertex in the vertical line AD. Draw a horizontal plane meeting the lines BD, CD, AD in the points E, F, G; then the angle EGF is the horizontal projection of $\angle BDC$, or is the $\angle BDC$ reduced to the horizon; and it is required to compute $\angle EGF$, supposing the angles BDC, BDA, CDA, determined instrumentally. Describe a sphere with center D and any radius, and let the planes EDF, FDG, GDE meet its surface in the arcs BC, CA, AB, forming the spherical triangle ABC; then its three sides are known, and the $\angle BAC = EGF$ is the angle sought, and may be computed from the formula

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}},$$

where $\angle BDC = a$, $\angle BDA = c$, $\angle CDA = b$, and $s = \frac{1}{2}(a + b + c)$.

58. In practice, the angles b and c usually differ so little from right angles, that an approximate formula will suffice.

Let $b = \frac{1}{2}\pi - h$, $c = \frac{1}{2}\pi - h'$, then making $\sin h = h$, $\cos h = 1 - \frac{1}{2}h^2$, and supposing $A = a + \theta$, where θ is very small, we have

$$\cos(a+\theta) = \frac{\cos a - \sin h \sin h'}{\cos h \cos h'} = \frac{\cos a - hh'}{1 - \frac{1}{2}(h^2 + h'^2)},$$

$$\therefore \cos a - \theta \sin a = (\cos a - hh')(1 + \tfrac{1}{2}h^2 + \tfrac{1}{2}h'^2)$$

$$= \cos a + \tfrac{1}{2}\cos a\,(h^2 + h'^2) - hh';$$

$$\therefore \theta = hh' \operatorname{cosec} a - \tfrac{1}{2}(h^2 + h'^2)\cot a.$$

Let $p = \frac{1}{2}(h + h')$, $q = \frac{1}{2}(h - h')$;

$$\therefore p^2 - q^2 = hh',\ p^2 + q^2 = \tfrac{1}{2}(h^2 + h'^2);$$

$$\therefore \theta = (p^2 - q^2)\operatorname{cosec} a - (p^2 + q^2)\cot a = p^2 \tan\tfrac{1}{2}a - q^2 \cot\tfrac{1}{2}a.$$

59. Given two sides and the included angle of a spherical triangle, to find the angle between the chords of those sides.

Let ABC (fig. 15) be a spherical triangle, O the center of the sphere, AB, AC, the chords of the arcs AB, AC, and a the angle included by them. Let pqr, be a spherical triangle described about A as center with any radius; then

$$\cos pq = \cos pr \cos qr + \sin pr \sin qr \cos prq.$$

But $\angle prq$ is the inclination of the planes AOB, AOC, and

$$= \angle A, \text{ and } pr = \angle BAO = 90^0 - \tfrac{1}{2}AOB = 90^0 - \tfrac{1}{2}c,$$

$$qr = \angle CAO = 90^0 - \tfrac{1}{2}AOC = 90^0 - \tfrac{1}{2}b;$$

$$\therefore \cos a = \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c + \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c \cos A.$$

Let $a = A - \theta$, where θ is usually very small; therefore, $\cos a = \cos A + \theta \sin A$; and the second member is the same as

$$\sin^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c) + \left\{\cos^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c)\right\}\cos A.$$

Hence, equating these values and reducing, we find

$$\theta = \sin^2\tfrac{1}{4}(b+c)\tan\tfrac{1}{2}A - \sin^2\tfrac{1}{4}(b-c)\cot\tfrac{1}{2}A, \text{ nearly.}$$

60. If in a spherical triangle the angle C and the opposite side c remain constant, to find the relations between the corresponding small variations of any two of the other parts.

First, to compare δa and δb the variations of the sides a and b. Writing down, in this and all similar cases, the formula involving the constant elements, and the two whose variations are to be compared, we have

$\cos c = \cos a \cos b + \sin a \sin b \cos C$;

$\therefore \cos c = \cos(a+\delta a)\cos(b+\delta b) + \sin(a+\delta a)\sin(b+\delta b)\cos C$;

or, since δa is very small, and therefore $\sin \delta a = \delta a$, $\cos\delta a = 1$, and the same for δb,

$$\cos c = (\cos a - \sin a\delta a)(\cos b - \sin b\delta b) + (\sin a + \cos a\delta a)$$
$$\times (\sin b + \cos b\delta b)\cos C;$$

therefore, subtracting, and neglecting terms involving $\delta a \delta b$,

$0 = \delta a(\sin a \cos b - \cos a \sin b \cos C) + \delta b(\sin b \cos a - \cos b \sin a \cos C)$,

or, dividing by $\sin a \sin b$, we get (Art. 22),

$$0 = \frac{\delta a}{\sin a}\cot B \sin C + \frac{\delta b}{\sin b}\cot A \sin C;$$

$$\therefore\ \delta a \cos B + \delta b \cos A = 0.$$

Hence from the polar triangle, if δA, δB be the variations of the angles, whilst C and c remain unchanged,

$$\delta A \cos b + \delta B \cos a = 0.$$

The above result may be also obtained geometrically, as follows.

Let ACB (fig. 26) be the triangle, and Ca, Cb, the altered values of the sides, so that $ab = AB$; draw the arcs $a\alpha$, $B\beta$ perpendicular to AO, bO, then $a\beta = B\alpha$ nearly, and therefore $A\alpha = b\beta$, or, since the triangles $Aa\alpha$, $Bb\beta$ may be regarded as plane triangles,

$$Aa \cos A = Bb' \cos Bb\beta, \text{ or } \delta a \cos B + \delta b \cos A = 0, \text{ nearly.}$$

To compare the variations of A and a, or of A and b, we must proceed in the same way with the equations

$$\sin A \sin c = \sin C \sin a,$$

$$\cot C \sin A = \cot c \sin b - \cos A \cos b;$$

and we shall find

$$\cot A\delta A = \cot a\delta a, \quad \cos B\delta A = -\cot a \sin A\delta b.$$

61. To compare the corresponding small variations of any two of the other parts of a triangle, in which two angles B, C, or two sides b, c, remain constant.

First, suppose B and C to remain constant; then proceeding as above with the equations

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$
$$\sin b \sin C = \sin c \sin B,$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$
$$\cot B \sin C = \cot b \sin a - \cos C \cos a,$$

we find

$$\delta A = \sin b \sin C \delta a, \ \cot b \delta b = \cot c \delta c, \ \sin A \delta b = \cot c \delta A,$$
$$\sin A \delta b = \sin B \cos c \delta a.$$

Hence by the polar triangle, supposing the two sides b and c to remain constant, we have

$$\delta a = \sin B \sin c \delta A, \ \cot B \delta B = \cot C \delta C,$$
$$\sin a \delta B = -\cot C \delta a, \ \sin a \delta B = -\sin b \cos C \delta A.$$

62. To compare the corresponding small variations of any two of the other parts of a triangle, in which an angle and a side adjacent to it A, c, remain constant.

Proceeding as above, we find

$$\delta a = \cos C \delta b, \ \tan a \delta C = -\tan C \delta a, \ \delta C = -\cos a \delta B,$$
$$\sin a \delta B = \sin C \delta b, \ \tan a \delta C = -\sin C \delta b, \ \sin a \delta B = \tan C \delta a.$$

Area of a spherical triangle. Radii of its inscribed and circumscribed circles. Regular Polyhedron, Parallelopiped, and Tetrahedron.

63. Given two sides and the included angle of a spherical triangle, to find the spherical excess.

$$-\cot \tfrac{1}{2} E = \tan \left\{\tfrac{1}{2}(A+B) + \tfrac{1}{2}C\right\} = \frac{\tan \frac{1}{2}(A+B) + \tan \frac{1}{2} C}{1 - \tan \frac{1}{2}(A+B) \tan \frac{1}{2} C}$$

$$= \frac{\dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2} C + \tan \frac{1}{2} C}{1 - \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}}.$$

$$= \frac{\cos\frac{1}{2}(a-b)\dfrac{1+\cos C}{\sin C} + \cos\frac{1}{2}(a+b)\dfrac{1-\cos C}{\sin C}}{\cos\frac{1}{2}(a+b) - \cos\frac{1}{2}(a-b)}$$

$$= \frac{1}{\sin C} \cdot \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b + \sin\frac{1}{2}a\sin\frac{1}{2}b\cos C}{-\sin\frac{1}{2}a\sin\frac{1}{2}b};$$

$$\therefore \cot\tfrac{1}{2}E = \cot\tfrac{1}{2}a\cot\tfrac{1}{2}b\operatorname{cosec} C + \cot C.$$

Hence the area of the triangle, which equals $\pi r^2 \times \dfrac{E}{180}$, is known.

64. Given three sides of a spherical triangle, to find the spherical excess.

$$\tan\tfrac{1}{4}E = \tan\tfrac{1}{2}\left\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^\circ - C)\right\}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(180^\circ - C)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(180^\circ - C)} \text{ (Pl. Trig. Art. 53)}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c} \cdot \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}, \text{ (Art. 25)}$$

$$= \frac{\sin\frac{1}{4}(a+c-b)\sin\frac{1}{4}(b+c-a)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)} \cdot \sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$= \sqrt{\left\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\right\}}.$$

65. To find the angular radius of the circle which passes through the angular points of a given triangle, in terms of its angles, or sides.

Draw the arcs aO, bO, (fig. 22), perpendicular to the sides BC, AC, through their middle points, and draw Oc perpendicular to AB. Join OA, OB, OC; then O is evidently the pole of the circumscribed circle, and c the middle point of AB. Hence $\angle C + \angle OAB = \frac{1}{2}(A+B+C)$, or $\angle OAB = S - C$; but from the right-angled triangle OAc, $\cos OAc = \cot OA \tan Ac$,

$$\therefore \tan R = \frac{\tan\frac{1}{2}c}{\cos(S-C)}.$$

Also, $\cos(S-C) = \cos\frac{1}{2}(A+B)\cos\frac{1}{2}C + \sin\frac{1}{2}(A+B)\sin\frac{1}{2}C$,

$$= \left\{\cos\tfrac{1}{2}(a+b) + \cos\tfrac{1}{2}(a-b)\right\}\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c} = \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b\sin C}{\cos\frac{1}{2}c},$$

$$\therefore \tan R = \frac{\sin\frac{1}{2}c}{\cos\frac{1}{2}a\cos\frac{1}{2}b\sin C},$$

and if for $\tan\frac{1}{2}c$, $\sin C$, we substitute their values (Arts. 48 and 36), we get $\tan R$ expressed in terms of the angles, and sides, respectively.

66. To find the angular radius of the circle which touches the three sides of a given triangle, in terms of its sides, or angles.

Bisect the angles A and B by the arcs AO, BO, (fig. 22), join CO, and from O draw the arcs Oa, Ob, Oc perpendicular to the sides, or sides produced; then these perpendiculars are evidently equal to one another, therefore O is the pole of the required circle, and the angle C is bisected by OC; also $Ac = Ab$, $Ba = Cc$ and $Ca = Cb$. Hence, when the circle is inscribed, $AB + Cb = \frac{1}{2}(a + b + c)$, or $Cb = s - c$; and when the circle touches the side c and the two others produced, $Cb = s$; but from the right-angled triangle CbO, $\sin Cb = \tan Ob \cdot \cot\frac{1}{2}C$;

$$\therefore\ \tan r = \sin(s - c)\tan\tfrac{1}{2}C,$$

for the inscribed circle; and for the circle which touches the side c, and the two others produced,

$$\tan r' = \sin s \tan\tfrac{1}{2}C.$$

Also $\tan r = \left\{\sin\frac{1}{2}(a + b)\cos\frac{1}{2}c - \cos\frac{1}{2}(a + b)\sin\frac{1}{2}c\right\}\tan\frac{1}{2}C$,

$$= \left\{\cos\tfrac{1}{2}(A - B) - \cos\tfrac{1}{2}(A + B)\right\}\frac{\sin\frac{1}{2}c\cos\frac{1}{2}c}{\cos\frac{1}{2}C}\ \text{(Art. 25)},$$

$$= \frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}\cdot\sin c;$$

$$\text{similarly,}\quad \tan r' = \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}\sin c;$$

and if for $\tan\frac{1}{2}C$, and $\sin c$, we substitute their values, we get $\tan r$ and $\tan r'$ expressed in terms of the sides, and angles, respectively.

67. If S be the number of the solid angles of a polyhedron, F the number of its faces, E the number of its edges, then $S + F = E + 2$.

From any internal point as a center, suppose a sphere to be described with radius equal to unity; join the center with

each of the angular points of the polyhedron, and then join all the points where these lines meet the surface of the sphere by arcs of great circles; there will thus be formed as many polygons as there are faces.

If therefore s be the sum of the angles of any one of these polygons, and n the number of its sides, (since it may be divided into as many triangles as it has sides, having a common vertex) its area $= s + 2\pi - n\pi$; and therefore, adding the areas of all the polygons together of which the number is F,

$$\text{area of sphere} = 4\pi = \Sigma\,(s) + 2F\pi - \pi\Sigma\,(n).$$

But $\Sigma\,(s)$ = sum of all the angles of all the polygons $= 2\pi \times$ number of solid angles $= 2\pi S$; and $\Sigma\,(n)$ = number of all the sides of all the polygons = twice the number of edges $= 2E$, because each edge gives an arc common to two polygons;

$$\therefore 4\pi = 2\pi S + 2\pi F - 2\pi E, \text{ or } S + F = E + 2.$$

68. There can be only five regular polyhedrons.

In the case of a regular polyhedron, every face has the same number (n) of sides, and every solid angle the same number (m) of faces; and the entire number of plane angles in all the faces is equally expressed by nF or mS or $2E$;

$$\therefore nF = mS = 2E, \text{ and } S + F = E + 2; \text{ hence}$$

$$S = \frac{4n}{2(m+n) - mn}, \text{ which must be a positive integer;}$$

$$\therefore \frac{1}{m} + \frac{1}{n} > \frac{1}{2}; \text{ but the greatest value both of } \frac{1}{m} \text{ and } \frac{1}{n} \text{ is } \frac{1}{3},$$

therefore neither $\frac{1}{m}$ nor $\frac{1}{n}$ can be so small as $\frac{1}{6}$; therefore the only admissible values for m and n are 3, 4, 5; and those combinations of them must be taken which make S, F, and E integers. It will be found that if the faces are equilateral triangles, or $n = 3$, we may form each solid angle of the polyhedron with 3, 4, or 5 angles of these triangles, and so form the tetrahedron, the octahedron, and the icosahedron, or the

solids of four, eight, and twenty faces, respectively. If the faces are squares, or $n = 4$, we may form each solid angle with three plane angles, and so form the cube or hexahedron; and if the faces are pentagons, or $n = 5$, by uniting three of their angles to form a solid angle, we obtain the dodecahedron; and these are all the regular polyhedrons that can exist.

69. To find the inclination of two adjacent faces of a regular polyhedron, and the radii of the inscribed and circumscribed spheres.

Let $AB = a$ (fig. 25) represent an edge of the polyhedron common to two adjacent faces whose centers are C and E; $CO = r$ the radius of the inscribed, $AO = R$ that of the circumscribed sphere; OD perpendicular to AB. Let the planes AOC, COD, AOD, meet the surface of a sphere whose center is O, in the arcs pq, qr, rp; then $\angle q = \frac{1}{2} \cdot \frac{2\pi}{n}$, $\angle p = \frac{2\pi}{2m}$,

$$\text{and} \quad \angle r = \tfrac{1}{2}\pi \, ; \quad \therefore \cos p = \cos qr \, . \sin q \, ;$$

$$\text{but} \quad \cos qr = \cos COD = \sin CDO = \sin \tfrac{1}{2} I, \text{ if } I = \angle CDE \, ;$$

$$\therefore \sin \tfrac{1}{2} I = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}} .$$

$$\text{Also} \quad \cos pq = \cot p \, . \cot q, \quad \text{or} \quad \frac{R}{r} = \tan \frac{\pi}{m} \tan \frac{\pi}{n} ,$$

$$\text{and} \quad R^2 = r^2 + \tfrac{1}{4} a^2 \operatorname{cosec}^2 \frac{\pi}{n} ,$$

from which equations R and r may be found.

The area of each face $= \frac{n}{4} a^2 \cot \frac{\pi}{n}$; (Pl. Trig. Art. 129)

$$\therefore \text{area of surface of polyhedron} = \frac{nF}{4} a^2 \cot \frac{\pi}{n} ,$$

and volume $= \frac{1}{3}$ area of surface $\times$ radius of inscribed sphere.

Also the radius of the sphere to which the edges are tan-

gents $OD = \sqrt{R^2 - \frac{1}{4}a^2}$; and that to which one face and the n adjacent ones produced, are tangents,

$$= CD \cot \tfrac{1}{2} I = \tfrac{1}{2} a \cot \frac{\pi}{n} \cot \tfrac{1}{2} I.$$

70. To find the volume of a parallelopiped in terms of its edges and their inclinations to one another.

Let the edges be $SA = a$, $SB = b$, $SC = c$, (fig. 27), and the angles which they make with one another $BSC = \alpha$, $ASC = \beta$, $ASB = \gamma$; drop the perpendicular CO from C on the plane ASB, and let the arc FG be the intersection of the plane CSO with the surface of a sphere whose center is S, and DE, EF, FD, the intersections of the faces of the parallelopiped with the same sphere. Then the volume of the parallelopiped

$= \text{area of base } AB \times \text{altitude } CO = ab \sin \gamma \times c \sin FG$

$= abc \sin \gamma \times \sin \alpha \sin E$, from the right-angled triangle FEG,

$= abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}$.

71. To find the diagonal of a parallelopiped in terms of its edges, and their inclinations to one another.

Let FH be the intersection of the plane SPT with the sphere whose center is S; then

$$ST^2 = SP^2 + PT^2 + 2SP \,.\, PT \cos FH$$

$$= a^2 + b^2 + 2ab\cos\gamma + c^2 + 2c \,.\, \frac{SP}{\sin\gamma} \{\cos\alpha \,.\, \sin HD + \cos\beta \sin HE\}$$

$$= a^2 + b^2 + c^2 + 2ab\cos\gamma + 2c\{b\cos\alpha + a\cos\beta\}, \text{ (Prob. 1).}$$

72. Given the six edges of any tetrahedron, to find its volume.

Let $SABC$ (fig. 27) be the tetrahedron; then it will evidently be one-third of the prism whose base is ASB and height CO, and consequently one-sixth of the parallelopiped whose edges are SA, SB, SC; and therefore its volume

$$= \tfrac{1}{6} abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma};$$

but if $AB = c'$, $AC = b'$, $BC = a'$, then $2ab \cos \gamma = a^2 + b^2 - c'^2$, $2ac \cos \beta = a^2 + c^2 - b'^2$, $2bc \cos \alpha = b^2 + c^2 - a'^2$; and these values of the cosines may be substituted.

PROBLEMS.*

1. In any triangle, to find the arc AD intercepted between a given point D in one of the sides, and the opposite angle, (fig. 8),

$$\cos AD = \cos AB \cos BD + \sin AB \sin BD \cos B,$$

and substituting for $\cos B$ its value in terms of the three sides, we get

$$\cos AD \sin BC = \cos AB \sin DC + \cos AC \sin BD.$$

2. On the surface of a sphere to draw a great circle passing through a given point and touching a given small circle.

Let B be the given point, (fig. 16), and P the pole of the given circle AC; then if BC be a great circle touching AC, and PC be joined, PCB is a right angle; and therefore $\cos BPC = \tan PC \cot PB$, which determines the point C, and consequently the circle BC.

3. If two arcs of great circles terminated by any circle, cut one another, the products of the tangents of the semi-segments are equal to one another.

Let AB, CD (fig. 17) be the two arcs which intersect in F, P the pole of the circle in which they terminate; join PC, PA, PF, and draw the perpendiculars PH, PG. Then

$$\cos PG = \frac{\cos PF}{\cos FG} = \frac{\cos PA}{\cos AG}, \quad \cos PH = \frac{\cos PF}{\cos FH} = \frac{\cos PC}{\cos CH};$$

$$\therefore \frac{\cos CH}{\cos FH} = \frac{\cos AG}{\cos FG}, \quad \frac{\cos CH - \cos FH}{\cos CH + \cos FH} = \frac{\cos AG - \cos FG}{\cos AG + \cos FG},$$

$$\text{or} \quad \tan \tfrac{1}{2} AF \,.\, \tan \tfrac{1}{2} FB = \tan \tfrac{1}{2} CF \,.\, \tan \tfrac{1}{2} FD.$$

4. The product of the sines of the semi-diagonals of a quadrilateral inscribed in a circle, is equal to the sum of the products of the sines of half the opposite sides.

Let the dotted lines (fig. 24) represent the chords of the arcs; then they all lie in the plane of the circle circumscribing the quadrilateral, and $AD \cdot BC = AB \cdot CD + AC \cdot BD$;

$\therefore \sin\frac{1}{2}AD . \sin\frac{1}{2}BC = \sin\frac{1}{2}AB . \sin\frac{1}{2}CD + \sin\frac{1}{2}AC . \sin\frac{1}{2}BD.$

It is easily seen that the sums of the two opposite angles of the quadrilateral are equal to one another.

5. If E and F be the middle points of the diagonals of any quadrilateral $ABCD$, then the sum of the cosines of its sides will equal $4\cos AE \cos BF \cos FE$.

From the triangles ABD, ACD, (fig. 24) we get by Prob. 1,

$$2\cos AE . \cos BE = \cos AB + \cos BD,$$

$$2\cos AE . \cos CE = \cos AC + \cos CD;$$

but $\cos BE = \cos BF . \cos FE + \sin BF . \sin FE \cos BFE$,

$\cos CE = \cos CF . \cos FE - \sin CF . \sin FE \cos BFE$;

$\therefore$ sum of cosines of the sides $= 4\cos AE \cos BF \cos FE$.

6. To prove that $\sin\frac{1}{2}$ (spherical excess) $= \dfrac{\sin\frac{1}{2}a \sin\frac{1}{2}b}{\cos\frac{1}{2}c}$ $\times \sin C$.

$$\sin\tfrac{1}{2}E = \sin\left\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^0 - C)\right\}$$

$$= \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C,$$

$$= \left\{\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\right\}\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c} \text{ (Art. 25)}$$

$$\frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\sin C.$$

Hence, also, (Art. 63)

$$\cos\tfrac{1}{2}E = \left\{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C\right\}\sec\tfrac{1}{2}c.$$

7. If two arcs QBA, Qba, (fig. 18), be intersected by two others PaA, PbB, in the points A, B, and a, b, then

$$\frac{\sin AQ}{\sin BQ} = \frac{\sin Aa}{\sin Pa}\,\frac{\sin Pb}{\sin Bb}.$$

For $\frac{\sin AQ}{\sin BQ} = \frac{\sin AQ}{\sin Aa} \cdot \frac{\sin Aa}{\sin Bb} \cdot \frac{\sin Bb}{\sin BQ}$.

$$= \frac{\sin a}{\sin Q} \cdot \frac{\sin Aa}{\sin Bb} \cdot \frac{\sin Q}{\sin b} = \frac{\sin Aa}{\sin Bb} \cdot \frac{\sin a}{\sin b}$$

$$= \frac{\sin Aa}{\sin Bb} \cdot \frac{\sin Pb}{\sin Pa}.$$

8. If two arcs ABC, abc (fig. 18) be intersected by three others which pass through the same point P, in the points A, B, C, and a, b, c, then

$$\frac{\sin Bb}{\sin Pb} \cdot \sin AC = \frac{\sin Aa}{\sin Pa} \sin BC + \frac{\sin Cc}{\sin Pc} \sin AB.$$

We have (p. 160)

$$\sin AC \cdot \sin BQ = \sin AQ \sin BC + \sin AB \sin CQ;$$

but $\frac{\sin AQ}{\sin CQ} = \frac{\sin Pc}{\sin Cc} \cdot \frac{\sin Aa}{\sin Pa}$, $\frac{\sin BQ}{\sin CQ} = \frac{\sin Pc}{\sin Cc} \frac{\sin Bb}{\sin Pb}$;

hence, eliminating $\sin BQ$ and $\sin AQ$ from the above equation, we get

$$\frac{\sin Bb}{\sin Pb} \cdot \sin AC = \frac{\sin Aa}{\sin Pa} \sin BC + \frac{\sin Cc}{\sin Pc} \sin AB.$$

This may be written

$$\left(\frac{\sin PB}{\tan Pb} - \cos PB\right) \sin AC = \left(\frac{\sin PA}{\tan Pa} - \cos PA\right) \sin BC$$

$$+ \left(\frac{\sin PC}{\tan Pc} - \cos PC\right) \sin AB; \text{ but (Prob. 1)},$$

$$\cos PB \sin AC = \cos PA \cdot \sin BC + \cos PC \sin AB,$$

$$\therefore \frac{\sin PB}{\tan Pb} \cdot \sin AC = \frac{\sin PA}{\tan Pa} \cdot \sin BC + \frac{\sin PC}{\tan Pc} \cdot \sin AB.$$

If P be the pole of AC, then

$$\tan Bb \cdot \sin AC = \tan Aa \cdot \sin BC + \tan Cc \cdot \sin AB,$$

which expresses the relation between the latitudes and longitudes of three places situated in the same great circle.

9. The arc passing through the middle points of the sides of any triangle, upon a given base, will meet the base produced, in a fixed point whose distance from the middle point of the base is a quadrant.

We have (fig. 18) $\dfrac{\sin AQ}{\sin CQ} = \dfrac{\sin Aa}{\sin Pa} \cdot \dfrac{\sin Pc}{\sin Cc}$ (Prob. 7),

if therefore a and c be the middle points of AP, CP, this becomes $\sin AQ = \sin CQ$; $\therefore AQ + CQ = 180^0$; and if B be the middle point of AC,

$$QB = \tfrac{1}{2}(AQ + CQ) = 90^0.$$

10. If one side of any triangle upon a given base be bisected by a secondary to the great circle which bisects the base at right angles, the other side will also be bisected by it.

For if $QB = 90^0$, then $AQ + QC = 180^0$, and $\sin AQ = \sin CQ$;

$$\therefore \frac{\sin Aa}{\sin Pa} = \frac{\sin Cc}{\sin Pc}, \quad \text{and if } Aa = Pa, \quad \text{than } Cc = Pc.$$

11. All triangles upon a given base, and which have the middle points of their other two sides in the same great circle, are equal in area. (fig. 19.)

Let AB be the given base, D its middle point, $DK = 90^0$, KFG any circle passing through K; then if ACB be a triangle having one of its sides AC bisected in G, the other will be bisected in F (Prob. 10); and if E be its spherical excess,

$$\cot \tfrac{1}{2} E = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} c + \cos B}{\sin B} = \cot K \operatorname{cosec} \tfrac{1}{2} c, \text{ from the}$$

triangle KBF (Art. 22), since $KD = 90^0$; $\therefore$ E is invariable. Also

$$\cos FG = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C = \cos \tfrac{1}{2} E \cos \tfrac{1}{2} c$$

(Prob. 6), therefore the distance of the middle points of the sides of all the triangles is the same.

12. Upon a given base to construct a triangle which shall have a given angle at the base, and shall be equal in area to a given triangle upon the same base.

AB (fig. 20) the given base, and ACB the given triangle, GF the arc joining the middle points of its sides.

Make BAG' = the given angle, take $G'F' = GF$, and produce BF' to meet AG' in C'; then ABC' is the triangle required. (Prob. 11.)

13. To construct a triangle which shall be equal in area to a given triangle, and have an angle in common with it, and have one of the sides containing that angle of a given length.

ACB the given triangle (fig. 21) and C the given angle. Take CB' equal to the given side, bisect BB' in F, and AB in G; produce FG to meet CA in F', make $F'A' = F'A$, and join $A'B'$; then $A'CB'$ is the required triangle. For the triangles $A'AB'$, ABB', are evidently equal to one another. (Prob. 11.)

14. To find the locus of the vertex of a spherical triangle of given base and area.

AB the given base (fig. 19), D its middle point, $Dp = x$, $pC = y$ the spherical co-ordinates of the vertex of the triangle whose area is E, α and β the areas of the right-angled triangles ACp, BCp, respectively;

$$\text{then} \quad \cot\tfrac{1}{2}\alpha = \cot(\tfrac{1}{4}c + \tfrac{1}{2}x)\cot\tfrac{1}{2}y, \quad \text{(Art. 63)}$$
$$\cot\tfrac{1}{2}\beta = \cot(\tfrac{1}{4}c - \tfrac{1}{2}x)\cot\tfrac{1}{2}y;$$

$$\therefore \cot\tfrac{1}{2}\alpha \,.\, \cot\tfrac{1}{2}\beta = \frac{\cos x + \cos\frac{1}{2}c}{\cos x - \cos\frac{1}{2}c}\cot^2\tfrac{1}{2}y, \quad \text{(Pl. Trig. Art. 53)},$$

$$\cot\tfrac{1}{2}\alpha + \cot\tfrac{1}{2}\beta = \frac{2\sin\frac{1}{2}c}{\cos x - \cos\frac{1}{2}c}\cot\tfrac{1}{2}y;$$

$$\therefore \cot\tfrac{1}{2}E = \frac{\cot\frac{1}{2}\alpha\cot\frac{1}{2}\beta - 1}{\cot\frac{1}{2}\alpha + \cot\frac{1}{2}\beta}$$
$$= \frac{\cos\frac{1}{2}c\operatorname{cosec}^2\frac{1}{2}y + \cos x(\cot^2\frac{1}{2}y - 1)}{2\sin\frac{1}{2}c\cot\frac{1}{2}y}$$
$$= \cot\tfrac{1}{2}c\operatorname{cosec}y + \cos x\operatorname{cosec}\tfrac{1}{2}c\cot y,$$

$$\text{or } \cos\tfrac{1}{2}c = \sin\tfrac{1}{2}c\cot\tfrac{1}{2}E\sin y - \cos x\cos y,$$
the equation to the required locus.

But if x', y', be the co-ordinates, reckoned from D, of the

pole, and r the angular radius of a circle of the sphere, the equation to its perimeter is evidently

$$\cos r = \sin y' \sin y + \cos y' \cos y \cos (x - x'),$$

which coincides with the above if

$$x' = 0, \ -\tan y' = \sin \tfrac{1}{2} c \cot \tfrac{1}{2} E, \ -\frac{\cos r}{\cos y'} = \cos \tfrac{1}{2} c\,;$$

and these three equations determine x', y', r.

The first shews that the center M of the locus lies in the secondary DD' to the base through its middle point; the second that $y' = \angle K + 90^0 = DE + 90^0$, GEF being an arc joining the middle points of the sides AC, BC, (Prob. 11); the third that if AC and BC be produced to meet the circle ADB in A', B', then A', B', are points in the locus; for cos $MD' = -\cos y'$, and $A'D' = \frac{1}{2}c$, therefore $\cos r = \cos MD' \cos A'D'$, consequently $r = MA' = MB'$. As the direct investigation of Lexell's Problem is rather intricate, it may be worth while to give the following easy proof of its converse.

15. If an arc be produced both ways till it equals the semi-circumference, and any small circle be described through the points to which it is produced; then any triangle having that arc for base, and its vertex in the small circle, will have a constant area.

Let AB (fig. 19) be the given arc produced to A', B'; M the pole of any small circle passing through A', B'; C any point in the small circle; join AC, BC, which being produced, must pass through A', B'. Then the sum of the angles of the triangle $ABC = AA'B + BB'A + A'CB'$

$$= MA'B + MB'A, \text{ since } MCA' = MA'C,\ MCB' = MB'C.$$

Hence the sum of the angles, and consequently the area, of triangle ACB, remains the same, for all positions of C in the small circle $A'CB'$.

16. If three arcs be drawn from the angles of a triangle through any point to meet the opposite sides, then the products of the sines of the alternate segments of the sides are equal to one another.

By Prob. 7, we have (fig. 23)

$$\frac{\sin BC'}{\sin BA} = \frac{\sin CB'}{\sin AB'} \cdot \frac{\sin C'P}{\sin CP}; \quad \frac{\sin AC'}{\sin AB} = \frac{\sin CA'}{\sin BA'} \cdot \frac{\sin C'P}{\sin CP};$$

$$\frac{\sin AC'}{\sin BC'} = \frac{\sin AB'}{\sin CB'} \cdot \frac{\sin CA'}{\sin BA'}.$$

Conversely, when three points A', B', C' in the sides satisfy this condition, the arcs joining them with the opposite angles intersect one another in the same point. The cases in which this condition is fulfilled are exactly the same as for plane triangles.

If A', B', C' be the feet of the perpendiculars dropped from the angles upon the opposite sides, the condition is evidently fulfilled, since we have from the properties of right-angled triangles

$$\cos AC' \cos CB' \cos BA' = \cos BC' \cos AB' \cos CA',$$

$$\tan AC' \tan CB' \tan BA' = \tan BC' \tan AB' \tan CA'.$$

17. If three arcs be drawn from the angles A, B, C of a triangle through any point P, to meet the opposite sides in A', B', C', then shall

$$\frac{\sin PA'}{\sin AA'} \cos PA + \frac{\sin PB'}{\sin BB'} \cos PB + \frac{\sin PC'}{\sin CC'} \cos PC = 1.$$

For if x, y, z, be the co-ordinates of P referred to OA, OB, OC as axes, then $x \cos POA + y \cos POB + z \cos POC = OP$ (1).

But drawing a line from P parallel to OA' to meet AO in Q, we get

$$\frac{x}{OP} = \frac{\sin OPQ}{\sin PQO} = \frac{\sin PA'}{\sin AA'},$$

and similarly of the rest; therefore by substituting in (1), we get the result stated.

18. If three arcs be drawn from the angles of a triangle through any point P, to meet the opposite sides in A', B', C', and M be the pole of the circumscribed circle, then

$$\frac{\sin PA'}{\sin AA'} + \frac{\sin PB'}{\sin BB'} + \frac{\sin PC'}{\sin CC'} \quad \frac{\cos PM}{\cos R}$$

Let O (fig. 23) be the center of the sphere, and let the radii OP, OM, OB', OA', meet the plane of the circumscribed circle in p, m, b, a; then $\angle Omp = 90^\circ$, and

$$\frac{pb}{Bb} = \frac{Op \sin PB'}{OB \sin BB'} = \frac{Op}{Om} \cdot \frac{Om}{OB} \cdot \frac{\sin PB'}{\sin BB'} = \frac{\cos BM}{\cos PM} \cdot \frac{\sin PB'}{\sin BB'};$$

$$\therefore \frac{pa}{Aa} + \frac{pb}{Bb} + \frac{pc}{Cc} =$$

$$1 = \left\{\frac{\sin PA'}{\sin AA'} + \frac{\sin PB'}{\sin BB'} + \frac{\sin PC'}{\sin CC'}\right\} \cdot \frac{\cos BM}{\cos PM}.$$

19. To find the angular distance between the poles of the inscribed and circumscribed circles of a triangle.

Suppose P (fig. 23) to be the pole of the inscribed circle; then from the right-angled triangle PNB'

$$\sin r = \sin PB' \sin B' = \frac{\sin PB'}{\sin BB'} \sin C \sin a,$$

$$\therefore \frac{\cos D}{\cos R \sin r} = \frac{1}{\sin B \sin c} + \frac{1}{\sin C \sin a} + \frac{1}{\sin A \sin b} = \frac{\sin a + \sin b + \sin c}{M},$$

where $M = \sin b \sin c \sin A$

$$= \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c},$$

$$\text{so that } \cot r = \frac{2 \sin s}{M}, \quad \tan R = \frac{4 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c}{M};$$

$$\therefore \left(\frac{\cos D}{\cos R \cos r}\right)^2 - 1$$

$$= \frac{2}{M^2}(1 + \sin a \sin b + \sin a \sin c + \sin b \sin c - \cos a \cos b \cos c)$$

$$= \frac{1}{M^2}\left\{2 \sin \tfrac{1}{2}(a + b) \cos \tfrac{1}{2} c + 2 \cos \tfrac{1}{2}(a - b) \sin \tfrac{1}{2} c\right\}^2 \text{ (p. 160)}$$

$$= \frac{1}{M^2}\left\{2 \sin s + 4 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c\right\}^2$$

$$= \left\{\cot r + \tan R\right\}^2;$$

$$\therefore \sin^2 D = \sin^2 (R - r) - \cos^2 R \sin^2 r.$$

If the circle touch the side c and the two others produced, then it will be similarly found that

$$\sin^2 D = \sin^2 (R + r') - \cos^2 R \sin^2 r'.$$

20. In a spherical triangle if the arcs of great circles bisecting two angles and terminated in the opposite sides, be equal, then the bisected angles shall be equal, provided their sum be less than two right angles.

Let $CD = BD'$ (fig. 13) then $\angle B = \angle C$; for if not, let $B > C$, and consequently $CD' > DB$; then also $CF > BF$, and $FD' > FD$; make therefore $FH = FB$ and $FG = FD$, and join GH; then the triangle FGH falls within $FD'C$, and therefore its area, or the sum of its angles, is less;

$$\therefore \angle G + \angle H = \angle D + \tfrac{1}{2} B < \angle D' + \tfrac{1}{2} C; \therefore \angle D' > \angle D.$$

Now conceive a triangle to be constructed on the other side of CB, with sides BE, CE, respectively equal to CD, BD; and $D'E$ to be joined. Then $BD'E = BED'$, $\therefore CD'E > CED'$, or $CE > CD'$, or $DB > CD'$; but DB is less than CD', which is absurd; consequently $\angle B = \angle C$. It will be observed that $\angle D'CE = ACB + ABC$ must be less than two right angles.

21. If O be the pole of the circle circumscribing a triangle ABC, then

$$\cos \tfrac{1}{2} AOB = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos C + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b.$$

This results from Art. 59; for it is evident that the angle between the chords of AC and BC is half the spherical angle AOB.

22. To find the radii r, R of the inscribed and circumscribed spheres of any tetrahedron.

First, we have $\frac{1}{3} r \times$ area of surface = volume. Also the center of the circumscribed sphere will evidently be the intersection of two lines at right angles to two faces passing through the centers of their circumscribed circles. Let EDF (fig. 28) be a plane bisecting the edge SC at right angles, and therefore containing the perpendiculars EO, FO, to the faces ACS, BSC, through the centers of the circles which circum-

scribe them, and consequently O the center of the circumscribed sphere. Then

$$SO^2 = SD^2 + DO^2 = SD^2 + \left(\frac{FE}{\sin D}\right)^2$$

$$= SD^2 + \frac{1}{\sin^2 D}(ED^2 + DF^2 - 2ED \,.\, DF \,.\, \cos D)$$

where $SD = \frac{1}{2}c$, $\quad ED = \dfrac{a - c\cos\beta}{2\sin\beta}$, $\quad DF = \dfrac{b - c\cos a}{2\sin a}$,

and $\quad \cos D = \dfrac{\cos\gamma - \cos a\cos\beta}{\sin a\sin\beta}$.

Or, if the length of an edge, the inclination of the faces which intersect in it, and the angles which it subtends at the other vertices of the tetrahedron, be given, then

$$SQ^2 = SD^2 + \frac{SD^2}{\sin^2 D}(\cot^2 A + \cot^2 B - 2\cot A\cot B\cos D).$$

23. To determine the arc which joins the vertices of two given triangles upon the same base.

Suppose the base bisected in O (fig. 20); then (Prob. 1)

$$2\cos\tfrac{1}{2}c\cos CO = \cos a + \cos b;$$

and if p be the perpendicular from C on AB, $\sin c\sin p = \sin a\sin b\sin C$; and similarly for the triangle $AC'B$. Hence

$$\cos CC' = \cos CO\cos C'O + \sin CO\sin C'O\cos(AOC' - AOC);$$

but $\sin CO\sin\frac{1}{2}c\cos AOC = \cos b - \cos\frac{1}{2}c\cos CO = \frac{1}{2}(\cos b - \cos a)$,

$$\sin CO\sin AOC = \sin p;$$

$$\therefore\ \sin^2 c \,.\, \cos CC' = (\cos a + \cos b)(\cos a' + \cos b')\sin^2\tfrac{1}{2}c$$
$$+ (\cos b - \cos a)(\cos b' - \cos a')\cos^2\tfrac{1}{2}c$$
$$+ \sin a\sin b\sin C\sin a'\sin b'\sin C';$$

which will involve only the sides, if for $\sin C$, $\sin C'$, their values be substituted. This is the Problem of finding the latitude from two altitudes of the Sun and the time between.

24. If M be the pole of the circle circumscribing an equilateral triangle ABC, and P any point on the sphere, then (fig. 23)

$$\cos PA + \cos PB + \cos PC = 3 \cos R \cos PM,$$

$$\cos PA = \cos R \cos D + \sin R \sin D \cos AMP,$$

$$\cos PB = \cos R \cos D + \sin R \sin D \cos (120^0 - AMP),$$

$$\cos PC = \cos R \cos D + \sin R \sin D \cos (120^0 + AMP);$$

$$\therefore \cos PA + \cos PB + \cos PC = 3 \cos R \cos D.$$

25. Having given the latitudes and longitudes of two places on the Earth's surface, to find their distance.

Let P be the pole (fig. 9) GQR the equator, A, B, the two places on the meridians PQ, PR; PG the meridian of Greenwich; then the difference of longitude $= GR - GQ = QR = \angle APB = C$ suppose, and the colatitudes $PA = b$, $PB = a$, are known; so that in the spherical triangle APB there are given two sides and the included angle to find the third side $AB = c$, which may be done by the formulæ of Art. 43,

$$\tan \theta = \tan a \cos C, \quad \cos c = \frac{\cos a \cos (b - \theta)}{\cos \theta};$$

then if D be the length of a quadrant of the meridian in miles, the distance of the places $= D \cdot \frac{c}{90}$.

26. From any formula in Spherical Trigonometry involving the parts of a triangle, one of them being a side, to deduce the corresponding formula in Plane Trigonometry.

Since the quantities involved are the angles of inclination of the faces and edges of a solid angle expressed by their circular measures, if we suppose α, β, γ to be the lengths of the arcs in which the planes of the faces cut the surface of a sphere described from the vertex of the solid angle with any radius r, we shall have $\alpha = ar$, $\beta = br$, $\gamma = cr$; and if we substitute for a, b, c these values in the proposed formula, and then suppose r to be very large so that $\frac{\alpha}{r}$, $\frac{\beta}{r}$, $\frac{\gamma}{r}$, become very small, and therefore $\sin \frac{\alpha}{r}$, $\cos \frac{\alpha}{r}$, &c. may be replaced

by one or two terms of their expansions, we obtain a nearly exact relation between the lengths of the arcs upon a sphere whose radius is r, and the angles of inclination of the planes in which the arcs lie; and if we now make r infinite, we obtain an exact relation between the sides and angles of a plane triangle. Thus, as at Art. 55, we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos\frac{a}{r} - \cos\frac{\beta}{r}\cos\frac{\gamma}{r}}{\sin\frac{\beta}{r}\sin\frac{\gamma}{r}}$$

$$= \frac{\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{1}{24}\frac{a^4}{r^4}\right) - \left(1 - \frac{1}{2}\frac{\beta^2}{r^2}\right)\left(1 - \frac{1}{2}\frac{\gamma^2}{r^2}\right)}{\frac{\beta}{r}\cdot\frac{\gamma}{r}}$$

$$= \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma} + \frac{a^4 + \&c.}{24\beta\gamma r^2} \text{ nearly, supposing } r \text{ very large};$$

$\therefore$ making r infinite, we get for a plane triangle whose sides are a, β, γ,

$$\cos A = \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma}.$$

FINIS.